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MAXIMAL α-R.E. SETS

BY

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ABSTRACT. Various generalizations of maximal sets from ordinary recursion theory to recursion theory on admissible ordinals are considered. A justification is given for choosing one of these definitions as superior to the rest. For all the definitions considered to be reasonable, a necessary and sufficient condition for the existence of such maximal α-r.e. sets is obtained.

0. Introduction. Let M be an r.e. subset of ω, the set of natural numbers. Call M *maximal* if $\omega - M$ is not finite, but given any r.e. set A, either $(\omega - A) \cap (\omega - M)$ is finite or $A \cap (\omega - M)$ is finite. In our generalizations of "maximal set" to recursion theory on an admissible ordinal α, we always replace "r.e." by "α-r.e.", but consider several notions in place of "finite". We then study sets maximal with respect to many of these generalizations of "finite". Our major result provides a necessary and sufficient condition on admissible ordinals α, for the existence of maximal α-r.e. sets for some of these definitions; maximal sets exist for some, but not all, countable admissible ordinals, but for no uncountable admissible ordinals. The precise conditions on α can be found at the end of §5. They are not given here because they require some auxiliary definitions not suitable for the introduction. The definition of "maximal" used does influence the existence result for sets maximal under that definition, but only in a very coarse way; the definition may impose the restriction that the complement of a maximal set have a short order type. But this restriction rules out the existence of maximal sets only if there also is no α-r.e., non-α-recursive set whose complement has an equally short order type.

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([2]) We would like to thank S. Simpson for his many helpful comments during and after the preparation of this paper. We would also like to express our sincere thanks to G. Kreisel for his criticism, especially relevant to our original introduction, of the superficial nature of our serious remarks and of the poor quality of our light remarks. We have expanded our original introduction, trying to make it less superficial and to discuss some points which Kreisel felt we had neglected. But we do not wish to muzzle Kreisel, so we have also expanded on our light remarks, maintaining the same poor quality.

We will study maximal sets in the setting of recursion theory on admissible ordinals.(3) Within this setting, we stress a lattice-theoretic approach. Let $\mathfrak{L}$ be the first order language of lattice theory. For each admissible ordinal α, let $\mathfrak{S}(\alpha)$ denote the lattice of α-r.e. sets, and let $\mathrm{Th}(\mathfrak{S}(\alpha))$ be the elementary theory of the lattice $\mathfrak{S}(\alpha)$. Thus we prefer to replace all occurrences of "finite" in the definition of maximal with α^*A-finite (X is α^*A-finite if every α-r.e. subset of X is α-recursive). Equivalently, X is α^*A-finite if X is α-finite and has order type less that α^*, where α^* is the least ordinal λ such that there is a one-one α-recursive function with domain α and range λ. We have adopted the lattice-theoretic point of view because we feel that the most significant known uses and properties of maximal sets are best formulated in terms of $\mathfrak{S}(\alpha)$. Lachlan [5] used their existence to prove the decidability of a certain class of sentences of $\mathfrak{L}$ in $\mathrm{Th}(\mathfrak{S}(\omega))$. Machtey [10] showed the same class of sentences of $\mathfrak{L}$ decidable in $\mathrm{Th}(\mathfrak{S}(\alpha))$ for all α for which maximal α-r.e. sets were known to exist at the time. It is hoped that this paper will yield some information towards the decidability of that same class of sentences of $\mathfrak{L}$ in $\mathrm{Th}(\mathfrak{S}(\alpha))$ for all admissible ordinals α. Soare [20] proved that given any two maximal ω-r.e. sets, there is an automorphism of $\mathfrak{S}(\omega)$ carrying one to the other. Consequently, any two maximal ω-r.e. sets have the same 1-type in $\mathfrak{L}$ over $\mathfrak{S}(\omega)$, i.e., any sentence of $\mathfrak{L}$ with one free variable satisfied by one maximal ω-r.e. set is satisfied by all maximal ω-r.e. sets. Owings [13] has shown that if $\alpha = \omega_1^{CK}$, i.e., α is the least admissible ordinal greater than ω, then there are at least two different 1-types of maximal α-r.e. sets over $\mathfrak{S}(\alpha)$ (hence there are maximal sets which are not automorphic to one another over $\mathfrak{S}(\alpha)$). We hope, however, that our work will begin to shed some light on the question of when two maximal α-r.e. sets are automorphic.

The generalizations of "X is finite" which we consider, involve restrictions both on the order type of the set X and on whether X is to be α-finite, α-r.e., or α-bounded. In choosing reasonable definitions of maximal set, we require that a set should not be maximal only because the order type of its complement is sufficiently small. Using such a definition, some maximal sets would be α-recursive (an uninteresting situation, in our opinion). These are excluded by our preferred notion of maximal set, which satisfies two basic criteria. First, the preferred class of maximal sets is definable over $\mathfrak{S}(\alpha)$. This fact was important in Lachlan's work quoted earlier. Second, the class is invariant under the following passage to a quotient lattice: Starting with the particular notion of finite used to

(3) We refer the reader who is interested in reasons for generalizing recursion theory to G. Kreisel, *Some reasons for generalizing recursion theory*, Logic Colloquium (Proceedings of the Summer School and Colloquium in Mathematical Logic, Manchester, August, 1969), North-Holland, Amsterdam, 1971, 139–198.

define maximal, we obtain a lattice by factoring $\mathcal{E}(\alpha)$ by the ideal consisting of these same generalized finite sets; a set is maximal with respect to the definition, if and only if it is a maximal element of this corresponding quotient lattice. Having determined to adopt a lattice-theoretic approach, this should be the justification for the use of the word maximal. The definition which we have chosen is the only one known to satisfy *both* these criteria. Our preferred definition is also broad enough to allow maximal sets with α-bounded complement, thus including Owings' example mentioned above. Yet it is narrow enough to prevent a set from being maximal solely because its complement is α-bounded (and thus inhibits the proliferation of 1-types of maximal sets). It remains to be seen whether or not the facts about $\mathcal{E}(\alpha)$ to be discovered in working with our definition can better be expressed by means of a different one. A generalization of Soare's automorphism theorem, mentioned above, to maximal α-r.e. sets could provide a good indication of the usefulness of our definition.

Our definitions of maximal α-r.e. sets include the nine definitions proposed by Kreisel and Sacks [3], all of which differ from our preferred definition. Sacks, in one of his jollier moods, related to me that Kreisel had originally wanted to propose twenty-seven definitions, and only with much difficulty was Sacks able to dissuade him from doing so. This was fortunate, else there would have been twenty-seven somewhat unsatisfactory definitions of a maximal α-r.e. set in [3].(4) We follow the somewhat dubious precedent set by Kreisel and Sacks in that we obtain a necessary and sufficient condition for the existence of a maximal α-r.e. set, not only for our preferred definition, but for all the definitions which we considered and decided were reasonable. Our justification for this is that little more work is involved, and the information obtained may prove to be of interest.

Maximal α-r.e. sets for some $\alpha > \omega$ were constructed by Kreisel and Sacks [3], and this construction was extended to a larger class of admissible ordinals by Lerman and Simpson [9]. These sets were maximal with respect to all the definitions in [3]. Sacks [16] showed that there are no maximal α-r.e. sets for $\aleph_1^L$ (the first cardinal which is not constructibly countable) under any definition of maximal α-r.e. sets in [3]; in fact, Sacks' proof works for any successor cardinal of L (Gödel's universe of constructible sets). Simpson [19] extended the class of admissible ordinals for which it was known that no maximal α-r.e. sets exist, and Lerman and Simpson [9] showed that there are no maximal α-r.e. sets when-

(4) The twenty-seven definitions were obtained by replacing each occurrence of "finite" in the definition of maximal set by one of {finite, α-finite, α-bounded}. Recently, A. Leggett has shown that if α-finite is used in all three places, the definition obtained is extensionally equivalent to our preferred definition. However, the α-finite sets do not give rise to a quotient lattice of $\mathcal{E}(\alpha)$, so this definition still does not satisfy our second criterion.

ever α is an uncountable admissible ordinal of L. We extend these positive and negative results to obtain a necessary and sufficient condition (on α) for the existence of a maximal α-r.e. set.

The outline of this paper is as follows. §1 deals with preliminary definitions and notation. §2 contains all the facts about α-recursion theory which we use, and which have nothing to do with maximal sets. In §3, we define various notions of a maximal α-r.e. set, give criteria for considering a definition reasonable, and explain why we feel that one definition is superior to the rest.

§4 contains the proof of the nonexistence of maximal α-r.e. sets for various α, and §5 contains the proof of the existence of maximal α-r.e. sets for various α. Combining the existence and nonexistence results, we obtain a necessary and sufficient condition for the existence of maximal α-r.e. sets. We conclude the paper in §6 with some remarks and open problems.

We have attempted to make this paper reasonably self contained, and thus hopefully accessible to all mathematical logicians. We assume only familiarity with Gödel's definition of the hierarchy of constructible sets, $\{L_\alpha: \alpha \text{ an ordinal}\}$, and the notion of a formula being Σ_n definable over L_α for all integers n. Several theorems are quoted but not proven. The proofs of these theorems follow almost immediately from definitions, with the exception of Jensen's Σ_n-projectum theorem [2]. One need only believe Jensen's result, however, to follow our proofs.

Certain abbreviations will be used in this paper. These abbreviations were recently ratified by the Cambridge consortium of α-recursion theorists, and will be defined in the paper.

1. Preliminaries. We first introduce the notation which we will be using.

Let α be an ordinal. If $A \subseteq \alpha$, then $\overline{A}$ will denote the relative complement of A in α, i.e. $\{x: x < \alpha \text{ and } x \notin A\}$. A will also be used to denote the characteristic function of A, i.e., that function defined by $A(x) = 1$ if $x \in A$ and $x < \alpha$, and $A(x) = 0$ if $x \notin A$ and $x < \alpha$.

If f is a function on ordinals and β is an ordinal, $f\upharpoonright_\beta$ will denote that function such that $f\upharpoonright_\beta(x) = f(x)$ if $f(x)$ is defined and $x < \beta$, and $f\upharpoonright_\beta(x)$ will be undefined otherwise.

If $f: \alpha \to \beta$ and $g: \beta \to \gamma$ are functions, then $g \circ f: \alpha \to \gamma$ will denote the function such that $g \circ f(x) = g(f(x))$ for all $x < \alpha$, if $f(x)$ and $g(f(x))$ are defined, and $g \circ f(x)$ will be undefined otherwise.

If f is a function, then $\mathrm{dom}(f)$ will denote the domain of f, i.e., $\{x: f(x) \text{ is defined}\}$. If f is a function of n variables, $\lambda x_1 \cdots x_k[f(x_1, \cdots, x_n)]$ will denote that function g of k variables such that $g(y_1, \cdots, y_k) = f(y_1, \cdots, y_k, x_{k+1}, \cdots, x_n)$ whenever the latter is defined, and $g(y_1, \cdots, y_k)$ will be undefined otherwise.

If f is a function, we say that $\lim_{\tau \to \sigma} f(\tau) = y$ if there is a λ with $\tau \leq \lambda < \sigma$

such that for all ν, if $\lambda \leq \nu < \sigma$, then $f(\nu)$ is defined and $f(\nu) = y$.

If γ and δ are ordinals, then $[\gamma]$ will denote $\{x: x < \gamma\}$, and $[\gamma, \delta]$ will denote $\{x: \gamma \leq x < \delta\}$. When confusion is unlikely, $[\gamma]$ and γ will be identified.

Let A be a set of ordinals. sup(A) will denote the least ordinal γ such that $x < \gamma$ for all $x \in A$. inf(A) will denote the least element of A. If A is finite, then max(A) will denote the greatest element of A.

The following definitions include the basic definitions of α-recursion theory.

Let α be an ordinal. A function f is said to be *partial α-recursive* if its graph is Σ_1 over L_α. f is said to be *α-recursive* if f is partial α-recursive, and dom$(f) = [\alpha]$. $A \subseteq \alpha$ is said to be *α-finite* if $A \in L_\alpha$. $A \subseteq \alpha$ is said to be *α-r.e.* if A is the range of an α-recursive function. $A \subseteq \alpha$ is said to be *α-recursive*, if both A and $\bar{A}$ are α-r.e. $A \subseteq \alpha$ is said to be *α-bounded* if there is a $\gamma < \alpha$ such that for all x, if $x \in A$, then $x < \gamma$. $A \subseteq \alpha$ is said to be *α-unbounded* if A is not α-bounded. $A \subseteq \alpha$ is said to be *γ-regular*, if $\gamma \leq \alpha$ and if for all $\beta < \gamma$, $A\restriction_\beta$ is α-finite. $A \subseteq \alpha$ is said to be *regular* if A is α-regular. Finally, we say that α is an *admissible* ordinal if given any partial α-recursive function f, and any α-finite set B such that $B \subseteq$ dom(f), then $f(B)$ is α-finite.

Since there is a one-one α-recursive correspondence between $[\alpha]$ and L_α, it suffices to consider only sets of ordinals and functions of ordinals in studying effective computability over L_α. This explains why the α-recursive functions were defined to be total only on $[\alpha]$, and not necessarily on all of L_α.

α will henceforth denote a fixed but arbitrary admissible ordinal. All functions should be considered to be total functions unless otherwise specified. Details about the mechanics of α-recursion theory can be found in [3] and [16]. We merely mention the following remarks, which will be used later in this paper without referring to this section.

Remark. Let A be an α-r.e. set which is not α-recursive. Then there is a one-one α-recursive function $f: \alpha \rightarrow A$ such that $f([\alpha]) = A$.

Remark. Let A be an α-recursive set which is not α-finite. Then there is a one-one α-recursive function $f: \alpha \rightarrow A$ enumerating the elements of A in order of magnitude.

Remark. Let A be an α-finite set. Then there is a $\gamma < \alpha$ and an α-finite function $f: \gamma \rightarrow A$ such that dom$(f) = [\gamma]$ and f enumerates the elements of A in order of magnitude.

Remark (Enumeration Theorem). There is an α-recursive enumeration $\{W_i: i < \alpha\}$ of all the α-r.e. sets. Furthermore, there is a double α-recursive enumeration $\{W_i^\sigma: i < \alpha$ and $\sigma < \alpha\}$ such that W_i^σ is α-finite for each $i < \alpha$ and $\sigma < \alpha$, $\bigcup\{W_i^\sigma: \sigma < \alpha\} = W_i$ for all $i < \alpha$, and if $\sigma < \tau < \alpha$, then $W_i^\sigma \subseteq W_i^\tau$ for all $i < \alpha$.

We will sometimes say that two quantities are equal when both are undefined.

Thus $a = b$ will mean that either a and b are both defined and equal, or both undefined.

2. **Cardinals, cofinalities, and projecta.** This section contains a potpourri of facts which will be used later. Many of these facts are used in most priority argument proofs in α-recursion theory, so we decided to separate them from the proofs of our main theorems, and organize them into a separate section.

We say that $\gamma < \alpha$ is an *α-cardinal* if there does not exist a one-one α-finite function with domain γ and range an ordinal $< \gamma$. γ is a *singular* α-cardinal if γ can be expressed as an α-finite union, over an α-finite index set of α-cardinality $< \gamma$, of α-finite sets, all of α-cardinality $< \gamma$. γ is a *regular* α-cardinal if γ is an α-cardinal which is not singular.

For any α-finite set A, let $\text{card}(A)$, the *α-cardinality* of A, be the least $\gamma < \alpha$ such that there is a one-one α-finite correspondence between A and γ. We note that for any α-finite set A, $\text{card}(A)$ is an α-cardinal.

The Σ_1 *projectum* of α, α^*, was introduced by Kripke [4], and is defined to be the least ordinal $\gamma \leq \alpha$ such that there is a one-one α-recursive function $f: \alpha \to \gamma$. Many of the following properties of α^* are well known.

Lemma 2.1. *If $\alpha^* < \alpha$, then α^* is the greatest α-cardinal.*

Proof. Clearly α^* is an α-cardinal. Let $f: \alpha \to \alpha^*$ be a one-one α-recursive function. If there were an α-cardinal $\beta > \alpha^*$, then since α is admissible, $f([\beta])$ would have α-cardinality $> \alpha^*$ since f is one-one, so $f([\beta]) \not\subseteq \alpha^*$. This contradiction proves the lemma.

Lemma 2.2. *Assume that $\alpha^* < \alpha$. Then there is an α-r.e. set $A \subseteq \alpha^*$ such that A is not α-recursive.*

Proof. Let $f: \alpha \to \alpha^*$ be a one-one α-recursive function. Then $f([\alpha])$ is an α-r.e. set. Since $f([\alpha])$ is bounded by α^*, if $f([\alpha])$ were α-recursive, $f([\alpha])$ would be α-finite. But then $f^{-1}: f([\alpha]) \to \alpha$ would be such that $f^{-1}(f([\alpha])) = [\alpha]$ which is not α-finite, contradicting the admissibility of α. Hence $f([\alpha])$ cannot be α-finite.

Lemma 2.3. *Assume that $\alpha^* < \alpha$. Let A be an α-r.e. set which is not α-recursive. Then there is an α-finite set $B \subseteq A$ such that B has order type $\geq \alpha^*$.*

Proof. Let f be a one-one α-recursive function enumerating A. Then $f([\alpha^*])$ is α-finite and has order type $\geq \alpha^*$.

Lemma 2.4. *Assume that $\alpha^* < \alpha$. Let A be an α-finite set such that A has* *order type* $\geq \alpha^*$. *Then there exists an* α*-r.e. set* $B \subseteq A$ *such that B is not* α*-recursive.*

Proof. By Lemma 2.2, there is an α-r.e. set $C \subseteq \alpha^*$ such that C is not α-recursive. Let f be a one-one α-recursive function enumerating C. Let $g: \alpha^* \to A$ enumerate the first α^* elements of A in order of magnitude. Let $B = g \circ f([\alpha])$. B is an α-r.e. set. If B were α-recursive, then since g is an α-finite function, B must be α-bounded, hence α-finite. But then, since α is admissible, $g^{-1}(B) = C$ would have to be α-finite, which is impossible by choice of C. Hence B cannot be α-recursive.

Lemma 2.5. *Let A be an* α*-r.e. set. Assume that A has order type* $\gamma < \alpha^*$. *Then A is* α*-finite.*

Proof. Assume that A is not α-finite. Then there is a one-one α-recursive function f enumerating A. But $f([\alpha^*]) \subseteq A$, and $f([\alpha^*])$ is α-finite and of order type $\geq \alpha^* > \gamma$, hence A cannot have order type γ. This contradiction proves the lemma.

Lemma 2.6. *Let A be an* α*-r.e. set which is not* α*-recursive. Let* $\gamma = \sup(\overline{A})$ *and let* β *be the order type of* $\overline{A}$. *Assume that* $\gamma < \alpha$. *Then* $\gamma \geq \alpha^*$ *and* $\beta \geq \alpha^*$.

Proof. A proof appears in [9]. We offer a slightly different proof. First assume for the sake of contradiction, that $\gamma < \alpha^*$. Then by Lemma 2.5, $\overline{A}\upharpoonright_\gamma$ is α-finite. Hence $\overline{A} = [\gamma] - A\upharpoonright_\gamma$ must be α-finite, so A must be α-recursive, contradicting the hypothesis of the lemma. We must therefore conclude that $\gamma \geq \alpha^*$.

We note that $A\upharpoonright_\gamma$ is α-r.e., but is not α-recursive, else $A = A\upharpoonright_\gamma \cup [\gamma, \alpha]$ would be α-recursive. Let $f: \alpha \to A\upharpoonright_\gamma$ be a one-one α-recursive function enumerating $A\upharpoonright_\gamma$. For each $s < \alpha$, let $A^s = \{y: (\exists x)(x < s \text{ and } f(x) = y)\}$. Then A^s is α-finite for each $s < \alpha$.

Without loss of generality, we can assume that A is γ-regular. For if not, and ν is the least ordinal such that $A\upharpoonright_{\nu+1}$ is not $\nu + 1$ regular, then A satisfies the hypothesis of the lemma with ν replacing γ, and $\beta' =$ the order type of $A\upharpoonright_\nu$, replacing β. Hence by the conclusion of the lemma, since $\beta' < \beta$, $\alpha^* \leq \beta' < \beta$.

For each $x < \gamma$, define $g(x)$ to be the least $s < \alpha$ such that $[x] - A^s$ has order type $< \beta$. Then g is a partial α-recursive function such that dom(g) is an initial segment of $[\gamma]$. Let $[\delta] = \text{dom}(g)$. Since α is admissible, $g([\delta])$ is α-finite. Let $\lambda = \sup(g([\delta]))$. Since $g([\delta])$ is α-finite, $\lambda < \alpha$. We note that $\gamma = \delta$. For if $\delta < \gamma$, since A is γ-regular, there is a $\lambda' < \alpha$ such that $A^{\lambda'}\upharpoonright_\delta = A\upharpoonright_\delta$; and since $\delta < \gamma = \sup(\overline{A})$, there is an $x \in \overline{A}$ such that $\delta < x < \gamma$, so $A\upharpoonright_\delta$ must have order type $< \beta$. Hence $[\delta] - A^{\lambda'}$ has order type $< \beta$, so $g(\delta)$ is defined. Since $\delta = \gamma$, by the definition of g and δ, $[\gamma] - A^{\lambda}$ has order type $\leq \beta$, and since $\lambda < \alpha$, $[\gamma] - A^{\lambda}$ is α-finite. Let $h: \beta \to [\gamma] - A^{\lambda}$ enumerate $[\gamma] - A^{\lambda}$ in order of magnitude. Let $p: \alpha \to \beta$ be defined by $p(x) = h^{-1} \circ f(\lambda + x)$. Then p is a one-one α-recursive function from α into β. By the definition of α^*, $\beta \geq \alpha^*$. This completes the proof of the lemma.

We will next define what we mean by an S_2 function. The reader will probably notice immediately that a function is S_2 over L_α if and only if it is Σ_2 over L_α, so we feel obligated to explain our motivation at this point. The definition of S_2 functions is more suitable for recursion theoretic purposes than the definition of Σ_2 functions. Furthermore, the definition gives rise to a natural hierarchy of functions, and it is in terms of S_3 functions that we get the necessary and sufficient condition for the existence of maximal α-r.e. sets. The set of S_3 functions does not seem to contain the set of Σ_3 functions, so the $\{\Sigma_n: n < \omega\}$ hierarchy is not suitable for our purposes. We note, however, that for $\alpha = \omega$, the $\{S_n: n < \omega\}$ hierarchy and the $\{\Sigma_n: n < \omega\}$ hierarchy coincide, so that no such distinction is necessary in ordinary recursion theory.

Let $\beta \leq \alpha$ and $\gamma \leq \alpha$. Let $f': \alpha \times \beta \to \gamma$ be an α-recursive function. We say that f' *generates an S_2 function* if for all $x < \beta$, $\lim_{\sigma \to \alpha} f'(\sigma, x)$ exists. $f: \beta \to \gamma$ is an S_2 *function* if there is an α-recursive function $f': \alpha \times \beta \to \gamma$ such that f' generates an S_2 function, and for all $x < \beta$, $f(x) = \lim_{\sigma \to \alpha} f'(\sigma, x)$. In this case, we say that f' *generates f as an S_2 function.*

If $\gamma \leq \alpha$ and $\delta \leq \alpha$, then $f: \gamma \to \delta$ is said to be a *cofinality function* if $f([\gamma])$ is cofinal with $[\delta]$.

Let $\beta \leq \alpha$ and $\gamma \leq \alpha$. Let $f: \beta \to \gamma$ be an S_2 function. We say that f is *tame*, if there is an α-recursive function $f': \alpha \times \beta \to \gamma$ such that f' generates f as an S_2 function and $(\delta < \beta)(\exists \sigma < \alpha)(\tau \geq \sigma)(x < \delta)(f'(\tau, x) = f(x))$, and $(\delta < \beta)$ $(\exists \lambda < \gamma)(f([\delta]) \subseteq [\lambda])$, i.e., f' witnesses the fact that $f\upharpoonright_\delta$ is α-finite and bounded below γ for all $\delta < \beta$. If $B \subseteq \beta \leq \alpha$, then we say that B *is a tame S_2 subset of β* if $B\upharpoonright_\beta$ (here B represents the characteristic function of B) is tame.

Let $\gamma \leq \alpha$. We define the S_2 *cofinality of γ*, s2cf(γ), to be the least ordinal $\delta \leq \gamma$ such that there exists an S_2 cofinality function $f: \delta \to \gamma$.

The S_2 cofinality of α was introduced by Stillwell [21] under the name recf(α). The S_2 cofinality of α has since appeared in print under many aliases, among them, the Σ_2 cofinality of α, Σ_2 − cof(α), and cf2(α). We use s2cf as the abbreviation recently adopted by the Cambridge consortium of α-recursion theorists. S_2 cofinality is important in α-recursion theory, because it tells you that if you simultaneously try to do less than s2cf(α) things α-effectively, and each thing involves $< \alpha$ steps, then the whole procedure is α-finite. The importance of this fact should be evident to those familar with the finite injury priority argument technique of ordinary recursion theory.

Tameness was introduced in [7], as a way to do Post's problem for admissible ordinals. It resulted from the observation that the Σ_1 substructure argument of Sacks and Simpson [17], despite its aesthetic appeal, was unnecessary to obtain a solution to Post's problem, and that the reason that the argument in [17] works, is that the indexing of requirements is tame. We noted in [7] that any sufficiently short tame indexing of requirements will suffice to do Post's problem for all admissible ordinals, and that it is unnecessary to construct one particular such indexing in advance. We will discuss the importance of tame functions in more detail after defining tame S_2 projecta. The definition of tame was introduced here in order to prove that there is a tame S_2 cofinality function $f\colon \mathrm{s2cf}(\gamma) \to \gamma$.

Lemma 2.7. *For all* $\gamma < \alpha$, $\mathrm{s2cf}(\gamma)$ *is an* α*-cardinal. If* $\mathrm{s2cf}(\alpha) < \alpha$, *then* $\mathrm{s2cf}(\alpha)$ *is an* α*-cardinal.*

Proof. Fix $\gamma \leq \alpha$, and assume $\mathrm{s2cf}(\alpha) < \alpha$ if $\gamma = \alpha$. Since $\mathrm{s2cf}(\gamma) \leq \gamma$, $\mathrm{s2cf}(\gamma) < \alpha$. Let $\delta = \mathrm{card}(\mathrm{s2cf}(\gamma))$, let $g\colon \delta \to \mathrm{s2cf}(\gamma)$ be a one-one α-finite correspondence between $\mathrm{s2cf}(\gamma)$ and δ, and let $f\colon \mathrm{s2cf}(\gamma) \to \gamma$ be an S_2 cofinality function. Then $f \circ g\colon \delta \to \gamma$ is an S_2 cofinality function, hence $\delta \geq \mathrm{s2cf}(\gamma)$, since $\delta = \mathrm{card}(\mathrm{s2cf}(\gamma)) \leq \mathrm{s2cf}(\gamma)$, the lemma is proven.

Lemma 2.8. *Let* $h\colon \delta \to \gamma$ *be an* S_2 *function where* δ *is a limit ordinal. If* h *is not tame, then* $\mathrm{s2cf}(\alpha) < \delta$.

Proof. Let $h'\colon \alpha \times \delta \to \gamma$ be an α-recursive function generating h as an S_2 function. Define $f'\colon \alpha \times \delta \to \alpha$ by $f'(0, x) = 0$ for all $x < \delta$, and $f'(\sigma, x) = \lim_{\tau \to \sigma} f'(\tau, x)$ if $\lim_{\tau \to \sigma} h'(\tau, y)$ exists for all $y \leq x$, and $f'(\sigma, x) = \sigma$ otherwise. Let μ be the least $\lambda < \delta$ such that either $\lim_{\sigma \to \alpha} f'(\sigma, \lambda)$ does not exist, or $f' \upharpoonright_{\alpha \times [\lambda+1]}$ does not generate a tame S_2 function. μ must exist since h is not tame. We now note that $f' \upharpoonright_{\alpha \times \mu}$ generates a tame S_2 function. Furthermore, if $f' \upharpoonright_{\alpha \times \mu}$ generates $f\colon \mu \to \alpha$ as an S_2 function, then f is a cofinality function, since for all $\lambda < \alpha$, there are $x < \mu$ and $\sigma \geq \lambda$ such that $h'(\sigma, x) \neq \lim_{\tau \to \sigma} h'(\tau, x)$, so $f'(\sigma, x) \geq \lambda$ for all $\sigma \geq \lambda$. Thus $\mathrm{s2cf}(\alpha) < \delta$.

Lemma 2.9. *For each* $\gamma \leq \alpha$, *there is a strictly increasing tame* S_2 *cofinality function* $g\colon \mathrm{s2cf}(\gamma) \to \gamma$.

Proof. There are two cases to consider.

Case 1. There is an α-recursive S_2 cofinality function $f\colon \mathrm{s2cf}(\gamma) \to \gamma$. We note that every partial α-recursive function with α-recursive domain is an S_2 function. Let $\beta = \mathrm{s2cf}(\gamma)$. For each $x < \beta$ define $g(x) = \sup(\{g(y)\colon y < x\} \cup \{f(x)\})$.

Then $\delta = \{x: g(x) < \gamma\}$ is an initial segment of β, so $\delta \leq \beta$. Furthermore, since $f(x) < \gamma$, $\delta = \beta$, else $g\restriction_\delta: \delta \to \gamma$ would be an S_2 cofinality function, contradicting the choice of $\beta = \mathrm{s2cf}(\gamma)$. g is strictly increasing by definition, and since every strictly increasing partial α-recursive function with α-recursive domain is tame, the lemma is proven in this case.

Case 2. No partial α-recursive function as in Case 1 exists. Let $\beta = \mathrm{s2cf}(\gamma)$ and let $f': \alpha \times \beta \to \gamma$ generate an S_2 cofinality function. We note that $\beta < \gamma \leq \alpha$, else the identity function on β would satisfy the conditions of Case 1. We define $g': \alpha \times \beta \to \gamma$ by induction on $\{\sigma: \sigma < \alpha\}$ and then by subinduction on $\{x: x < \beta\}$. If $\lim_{\tau \to \sigma} f'(\tau, y)$ exists for all $y \leq x$, define $g'(\sigma, x) = \lim_{\tau \to \sigma} g'(\tau, x)$. Otherwise, define $g'(\sigma, x) = \sup(\{g'(\sigma, y): y < x\} \cup \{f'(\sigma, y): y < \beta\})$.

We first note that if $\lim_{\tau \to \sigma} f'(\tau, y)$ exists for all $y \leq x$, then $\lim_{\tau \to \sigma} g'(\tau, y)$ exists for all $y \leq x$, since $g'(\tau, x)$ can change its value only if $f'(\tau, y)$ changes its value for some $y \leq x$. $\sup(\{g'(\sigma, y): y \leq x\}) < \gamma$ for all $x < \beta$ and $\sigma < \alpha$, else $\lambda y g'(\sigma, y): [x] \to \gamma$ is an S_2 cofinality function, which is impossible since $x < \beta = \mathrm{s2cf}(\gamma)$. Also, $\sup(\{f'(\sigma, y): y < \beta\}) < \gamma$ for all $\sigma < \alpha$, else $\lambda y f'(\sigma, y): \beta \to \gamma$ would satisfy the condition for Case 1. Hence g' is well defined, and total on $\alpha \times \beta$.

Assume that g' does not generate a tame S_2 function. Let δ be the least $\lambda < \beta$ such that either $\lim_{\sigma \to \alpha} g'(\sigma, \beta)$ is not defined, or $g'\restriction_{\alpha \times [\beta+1]}$ does not generate a tame S_2 function. Let $h': \alpha \times \delta \to \gamma$ be defined by $h'(\sigma, x) = g'(\sigma, x)$ for all $\sigma < \alpha$ and $x < \delta$. Then h' generates a tame S_2 function, and for all $\sigma < \alpha$ there are τ and x such that $\sigma \leq \tau < \alpha$ and $x < \delta$ and $f'(\tau, x) \neq \lim_{\lambda \to \tau} f'(\lambda, x)$. Let $\nu < \gamma$ be given, and choose $z < \beta$ such that $f(z) \geq \nu$. Let σ be such that for all $\tau \geq \sigma$, $f'(\tau, z) = f(z)$. Then for some $x < \delta$, there is a $\tau \geq \sigma$ such that $f'(\tau, x) \neq \lim_{\lambda \to \tau} f'(\lambda, x)$. Fix such an x, and a τ for x. Then for all $\lambda \geq \tau$, $g'(\lambda, x) \geq f(z) \geq \nu$, so $h'(\lambda, x) \geq \nu$. Thus h' generates an S_2 cofinality function, contradicting the choice of $\delta < \beta = \mathrm{s2cf}(\gamma)$. We must therefore conclude that g' generates a tame S_2 cofinality function g.

Since $\lambda x g'(\sigma, x)$ is strictly increasing, g must be strictly increasing, so the lemma is proved.

Lemma 2.10. *Let A be an α-r.e. set which is not α-recursive, and let β be the order type of $\overline{A}$. Then* $\mathrm{s2cf}(\alpha) \leq \beta$.

Proof. Let f be a one-one α-recursive function enumerating A. Let $A^\sigma = \{x: (\exists y)(y < \sigma \text{ and } f(x) = y)\}$. Let $\{a_i^\sigma: i < \beta\}$ enumerate the first β elements of $\overline{A}^\sigma$ in order of magnitude, and let $\{a_i: i < \beta\}$ enumerate $\overline{A}$ in order of magnitude.

We define an α-recursive function $g': \alpha \times \beta \to \alpha$ as follows: If $\sigma = 0$ and $x < \beta$, then $g'(\sigma, x) = 0$; if $\sigma > 0$ and $x < \beta$, then if $a_x^\sigma \neq \lim_{\tau \to \sigma} a_x^\tau$, we define $g'(\sigma, x) = \sigma$; and if $a_x^\sigma = \lim_{\tau \to \sigma} a_x^\tau$, then $g'(\sigma, x) = \nu$, where ν is the least ordinal such that $a_x^\nu = a_x^\sigma$.

For each $x < \beta$, let $S_x = \{\sigma: g'(\sigma, x) \neq \lim_{\tau \to \sigma} g'(\tau, x)\}$. Let S_x have order type μ and let $\{s(i, x): i < \mu\}$ be the enumeration of the elements of S_x in order of magnitude. Note that S_x is α-recursive. Assume that S_x is α-unbounded, for the sake of obtaining a contradiction. Then $\mu = \alpha$, and for all $i < \alpha$, $g'(s(i, x), x) \neq \lim_{\tau \to s(i,x)} g'(\tau, x)$ for every x. By the definition of g', we must have $a_i^{s(i,x)} \neq \lim_{\tau \to s(i,x)} a_i^\tau$ for all $i < \alpha$ and $x < \beta$. But if $\sigma < \tau$ then $a_x^\sigma < a_x^\tau$, so the function $h_x: \alpha \to \alpha$ defined by $h_x(\sigma) = a_x^{s(\sigma,x)}$ is a strictly increasing α-recursive function for each $x < \beta$, hence the range of h_x must be α-unbounded for each such x. But for all σ, $a_x^\sigma < a_x < \alpha$, yielding a contradiction. We must therefore conclude that S_x is α-bounded for each $x < \beta$. Hence $\lim_{\tau \to \alpha} g'(\tau, x)$ must exist for each $x < \beta$, so g' generates an S_2 function $g: \beta \to \alpha$.

Let $\lambda < \alpha$ be given. Since A is not α-recursive, there exists a $\delta > \lambda$ such that $f(\delta) = a_x^\lambda$ for some $x < \beta$. Fix the least such δ. Then $\lim_{\tau \to \delta} a_x^\tau \neq a_x^\delta$ for this x. Hence $g'(\sigma, x) \geq \delta > \lambda$ for all $\sigma \geq \delta$. Thus $g(x) > \lambda$, so g is a cofinality function. Hence $\beta \geq \mathrm{s2cf}(\alpha)$. This concludes the proof of the lemma.

Lemma 2.11. *Let A be an α-r.e. set which is not α-recursive, and assume that $\overline{A} \subseteq \gamma < \alpha$. If* $\mathrm{card}(\gamma) = \omega$, *then* $\mathrm{s2cf}(\alpha) = \omega$.

Proof. Since $\overline{A} \subseteq \gamma$, $\overline{A}$ has order type $\leq \gamma$, so by Lemma 2.10, $\mathrm{s2cf}(\alpha) \leq \gamma$. Since α is admissible, $\mathrm{s2cf}(\alpha) \geq \omega$. Since $\mathrm{card}(\gamma) = \omega$ and $\mathrm{s2cf}(\alpha) \leq \gamma$, $\mathrm{s2cf}(\alpha) \leq \omega$ by Lemma 2.7. The lemma now follows immediately.

We will next define the tame S_2 projectum of γ, $\mathrm{ts2p}(\gamma)$, for all $\gamma < \alpha$. We introduced $\mathrm{ts2p}(\alpha)$ in [7] as $\mathrm{tp2}(\alpha)$, the tame Σ_2 projectum of α, and used it in [8] under a different, but equivalent definition. The change from tp2 to ts2p follows the recommendations of the Cambridge consortium of α-recursion theorists. The definition we will give below is different from the previous definitions, but is equivalent to them. The change in definition follows the recommendation of S. Simpson, who feels that it is more natural to look at maps going up than at maps going down, and that functions being considered no longer need be one-one. We completely agree with him, and note also that this change simplifies notation.

We say that $f: \delta \to \gamma$ is a *projection* if $f([\delta]) = [\gamma]$. For each $\gamma \leq \alpha$, we define $\mathrm{ts2p}(\gamma)$, *the tame S_2 projectum of* γ, to be the least ordinal $\delta \leq \gamma$ such that there is a tame S_2 projection $f: \delta \to \gamma$.

The importance of $\mathrm{ts2p}(\alpha)$ in priority arguments is that it allows a short indexing of all requirements such that every proper initial segment of requirements under this indexing reaches its final set of priorities at some α-finite stage. The shortness of the indexing eases convergence problems. We use ts2p(α) in a different way in §4; it enables us to get S_3 projections from ν onto ω, uniformly for all $\nu <$ ts2p(α).

Lemma 2.12. s2cf(α) $\leq$ ts2p(α) $\leq \alpha^*$.

Proof. The fact that s2cf(α) $\leq$ ts2p(α) is immediate from the definitions. The proof that ts2p(α) $\leq \alpha^*$ can be found in [8]. Briefly, it goes as follows.

Let f: $\alpha \rightarrow \alpha^*$ be a one-one α-recursive function, and let $A = f([\alpha])$. Let β be the order type of A, and let g: $\beta \rightarrow A$ be the one-one, onto, order preserving function. Then $f^{-1} \circ g$: $\beta \rightarrow \alpha$ is a tame S_2 projection.

Lemma 2.13 (Simpson). ts2p(α) *is at least* δ *such that not every tame* S_2 *subset of* δ *is* α-*finite.*

Proof. We will only need the lemma to show that if δ is an α-cardinal and there is a tame S_2 subset I of δ which is not α-finite, then ts2p(α) $\leq \delta$. Since the proof of the lemma has not yet appeared in print, we offer a sketchy proof of this latter fact.

We assume that $\delta < \alpha$, else the lemma is immediate. Since I is an S_2 subset of δ, there is an α-recursive function f: $\alpha \times \delta \rightarrow \{0, 1\}$ such that $I(x) = \lim_{\sigma \rightarrow \alpha} f(\sigma, x)$ for all $x < \delta$. I is a tame S_2 subset of δ which is not α-finite, so the function g: $\delta \rightarrow \alpha$ defined by $g(x)$ is the least σ such that $f(\tau, x) = f(\sigma, x)$ for all $\tau \geq \sigma$, is a tame S_2 cofinality function. S_2 and Σ_2 are identical, so by Jensen's Σ_2 projectum theorem [1] there is a partial S_2 function h mapping a subset A of δ onto α. Since h is S_2 on A, there is an α-recursive function h': $\alpha \times \delta \rightarrow \alpha$ such that for all $x \in A$, $h(x) = \lim_{\sigma \rightarrow \alpha} h'(\sigma, x)$. For $x < \delta$ and $y < \delta$, define p: $\delta \cdot \delta \rightarrow \alpha$ by $p(\delta \cdot x + y) = h'(g(x), y)$. Since $\delta < \alpha$ and g is tame, p is a tame S_2 projection. Let q: $\delta \rightarrow \delta \cdot \delta$ be a tame α-finite projection. q exists since δ is an α-cardinal. Then $p \circ q$: $\delta \rightarrow \alpha$ is a tame S_2 projection, hence ts2p(α) $\leq \delta$.

We are now ready to introduce S_3 functions. Let f': $\alpha \times \alpha \times \beta \rightarrow \gamma$ be an α-recursive function. We say that f' *generates an* S_3 *function* if for all $\sigma < \alpha$, $\lim_{\tau \rightarrow \alpha} f'(\sigma, \tau, x)$ exists for all $x < \beta$, and $\lim_{\sigma \rightarrow \alpha} \lim_{\tau \rightarrow \alpha} f'(\sigma, \tau, x)$ exists for all $x < \beta$. We call f: $\beta \rightarrow \gamma$ an S_3 *function*, if there is an α-recursive function f': $\alpha \times \alpha \times \beta \rightarrow \gamma$ such that f' generates an S_3 function and for all $x < \beta$, $f(x) = \lim_{\sigma \rightarrow \alpha} \lim_{\tau \rightarrow \alpha} f'(\sigma, \tau, x)$. In this case, we say that f' *generates* f *as an* S_3 *function.*

Lemma 2.14. *If* f: $\beta \rightarrow \gamma$ *is an* S_2 *function, then* f *is an* S_3 *function.*

Proof. Let f': $\alpha \times \beta \to \gamma$ generate f as an S_2 function. Define g': $\alpha \times \alpha \times \beta \to \gamma$ by $g'(\sigma, \tau, x) = f'(\tau, x)$. Clearly g' generates f as an S_3 function.

We define s3cf(γ), *the* S_3 *cofinality of* γ, for each $\gamma \leq \alpha$ by s3cf(γ) is the least ordinal δ such that there is an S_3 cofinality function f: $\delta \to \gamma$.

Lemma 2.15. *For all* $\gamma \leq \alpha$, s3cf(γ) $\leq$ s2cf(γ).

Proof. Immediate from the definitions and Lemma 2.14.

Lemma 2.16. *For all* $\gamma \leq \alpha$, s3cf(γ) *is an* α*-cardinal, whenever* s3cf(γ) $< \alpha$.

Proof. As in the proof of Lemma 2.7.

We define s3p(γ), *the* S_3 *projectum* of γ, for each $\gamma \leq \alpha$ by s3p(γ) is the least ordinal δ such that there is an S_3 projection f: $\delta \to \gamma$.

Lemma 2.17. *For all* $\gamma \leq \alpha$, s3p(γ) $\leq$ ts2p(γ).

Proof. Immediate from the definitions and Lemma 2.14.

Lemma 2.18. *For all* $\gamma \leq \alpha$, *if* s3p(γ) $< \alpha$, *then* s3p(γ) *is an* α*-cardinal.*

Proof. As in the proof of Lemma 2.7.

Lemma 2.19. *Let* f: $\beta \to \gamma$ *and* g: $\gamma \to \delta$ *be* S_3 *functions. Then* $g \circ f$: $\beta \to \delta$ *is an* S_3 *function. If, furthermore,* f *and* g *are projections, then* $g \circ f$ *is a projection.*

Proof. The second half of the theorem is immediate.

Let f': $\alpha \times \alpha \times \beta \to \gamma$ generate f as an S_3 function, and let g': $\alpha \times \alpha \times \gamma \to \delta$ generate g as an S_3 function. Define h': $\alpha \times \alpha \times \beta \to \delta$ by $h'(\sigma, \tau, x) = g'(\sigma, \tau, f'(\sigma, \tau, x))$. Then $\lim_{\sigma \to \alpha} \lim_{\tau \to \alpha} h'(\sigma, \tau, x) = g(f(x))$, so h' generates $g \circ f$ as an S_3 function.

Lemma 2.20. s3cf(α^*) = s3cf(α).

Proof. We may assume that $\alpha^* < \alpha$. Let f: $\alpha \to \alpha^*$ be an α-recursive function. We note that $A = f([\alpha])$ is an α-r.e. set. A cannot be α-recursive, else f^{-1}: $A \to \alpha$ would be a partial α-recursive function and A an α-finite set, contradicting the admissibility of α since $f^{-1}(A) = [\alpha]$.

Let g: s3cf(α) $\to \alpha$ be an S_3 cofinality function. Then $f \circ g$: s3cf(α) $\to \alpha^*$ is an S_3 function. If sup($f \circ g$([s3cf(α)])) = α^*, then s3cf(α^*) $\leq$ s3cf(α). But if sup($f \circ g$([s3cf(α)])) = $\delta < \alpha^*$, then by Lemma 2.5, $A \cap [\delta]$ is α-finite, so since α is admissible, $f^{-1}(A \cap [\delta])$ must be α-bounded. But g([s3cf(α)]) $\subseteq f^{-1}(A \cap [\delta])$ contradicting the fact that g is a cofinality function. Hence s3cf(α^*) $\leq$ s3cf(α).

Let h: s3cf(α^*) $\to \alpha^*$ be an S_3 cofinality function, and let

$h': \alpha \times \alpha \times \mathrm{s3cf}(\alpha^*) \to \alpha^*$ be an α-recursive function generating h as an S_3 function. Define $p': \alpha \times \alpha \times \mathrm{s3cf}(\alpha^*) \to \alpha$ by $p'(\sigma, \tau, x) = \sup(\{\lambda: \lambda < \tau$ and $f(\lambda) < h'(\sigma, \tau, x)\})$. Since $h'(\sigma, \tau, x) < \alpha^*$, and since by Lemma 2.5, $\{\lambda: \lambda < \tau$ and $f(\lambda) < h'(\sigma, \tau, x)\}$ is α-finite, $\lim_{\tau \to \alpha} p'(\sigma, \tau, x)$ exists for all $\sigma < \alpha$ and $x < \mathrm{s3cf}(\alpha^*)$. Furthermore, $\lim_{\sigma \to \alpha}\lim_{\tau \to \alpha} p'(\sigma, \tau, x) = \sup(\{\lambda: \lambda < \alpha$ and $f(\lambda) < h(x)\})$ so p' generates an S_3 function p. Since $h([\mathrm{s3cf}(\alpha^*)])$ is cofinal with α^* and since each $\lambda < \alpha$ is such that $f(\lambda) < h(x)$ for some $x < \mathrm{s3cf}(\alpha^*)$, $p: \mathrm{s3cf}(\alpha^*) \to \alpha$ is an S_3 cofinality function. Hence $\mathrm{s3cf}(\alpha) \leq \mathrm{s3cf}(\alpha^*)$. This completes the proof of the lemma.

Lemma 2.21. *For all* $\gamma \leq \alpha$, $\mathrm{s3cf}(\mathrm{ts2p}(\gamma)) \leq \mathrm{s3cf}(\gamma)$.

Proof. Let $f: \mathrm{s3cf}(\gamma) \to \gamma$ be an S_3 cofinality function, and let $g: \mathrm{ts2p}(\gamma) \to \gamma$ be a tame S_2 projection. Let $f': \alpha \times \alpha \times \mathrm{s3cf}(\gamma) \to \gamma$ generate f as an S_3 function, and let $g': \alpha \times \mathrm{ts2p}(\gamma) \to \gamma$ generate g as an S_2 function. We define an α-recursive function $h': \alpha \times \alpha \times \mathrm{s3cf}(\gamma) \to \mathrm{ts2p}(\gamma)$ as follows. $h'(\sigma, \tau, x) = y$ if $f'(\sigma, \tau, x) = z$ and y is the least ordinal $< \tau$ such that $g'(\tau, y) = z$ if such an ordinal exists, and $y = 0$ otherwise.

Since g is a tame S_2 projection, $\lim_{\sigma \to \alpha}\lim_{\tau \to \alpha} h'(\sigma, \tau, x)$ must exist for all $x < \mathrm{s3cf}(\gamma)$. Hence h' generates an S_3 function $h: \mathrm{s3cf}(\gamma) \to \mathrm{ts2p}(\gamma)$. If $\sup(h([\mathrm{s3cf}(\gamma)])) = \delta < \mathrm{ts2p}(\gamma)$, then since $f([\mathrm{s3cf}(\gamma)]) \subseteq g([\mathrm{ts2p}(\gamma)] \cap [\delta])$ and since g is tame, $\sup(f([\mathrm{s3cf}(\gamma)])) \leq \sup(g([\mathrm{ts2p}(\gamma) \cap [\delta])) = \lambda < \gamma$, so f cannot be a cofinality function. We must therefore conclude that $\sup(h([\mathrm{s3cf}(\gamma)])) = \mathrm{ts2p}(\gamma)$, hence $\mathrm{s3cf}(\mathrm{ts2p}(\gamma)) \leq \mathrm{s3cf}(\gamma)$.

Lemma 2.22. *Let δ be a limit ordinal, and let $f': \alpha \times \alpha \times \omega \times \delta \to \delta$ be an α-recursive function such that for each $\nu < \delta$, $\lambda\sigma\tau x f'(\sigma, \tau, x, \nu)$ generates an S_3 projection $f_\nu: \omega \to \nu$. Assume furthermore, that* $\mathrm{s3cf}(\delta) = \omega$. *Then* $\mathrm{s3p}(\delta) = \omega$.

Proof. Since δ is a limit ordinal, $\mathrm{s3p}(\delta) \geq \omega$. We will define an α-recursive function $h': \alpha \times \alpha \times \omega \cdot \omega \to \delta$ such that h' generates an S_3 projection. Hence $\mathrm{s3p}(\delta) \leq \omega \cdot \omega$. By Lemma 2.18, we can then conclude that $\mathrm{s3p}(\delta) = \omega$.

Let $g': \alpha \times \alpha \times \omega \to \delta$ generate the S_3 cofinality function g. Define $h'(\sigma, \tau, \omega \cdot n + m) = f'(\sigma, \tau, m, g'(\sigma, \tau, n))$ for all $\sigma < \alpha$, $\tau < \alpha$, $m < \omega$, and $n < \omega$. h' is clearly α-recursive.

Since f' and g' generate S_3 functions we must have $\lim_{\tau \to \alpha} h'(\sigma, \tau, x)$ exists for all $\sigma < \alpha$ and $x < \omega \cdot \omega$. Fix $m < \omega$ and $n < \omega$. Then since f' and g' generate S_3 functions f and g, $\lim_{\sigma \to \alpha}\lim_{\tau \to \alpha} h'(\sigma, \tau, \omega \cdot n + m) = f_{g(n)}(m)$. Hence h' generates an S_3 function $h: \omega \cdot \omega \to \delta$ such that for all $m < \omega$ and $n < \omega$, $h(\omega \cdot n + m) = f_{g(n)}(m)$. Since g is a cofinality function, and since for all $n < \omega$, $f_{g(n)}: \omega \to [g(n)]$ is a projection, h must be a projection. This concludes the proof of the lemma.

3. Definitions of maximal sets. Let $M \subseteq \omega$ be an ω-r.e. set. We say that M is maximal if $\overline{M}$ is not finite, but given any ω-r.e. set C, either $\overline{M} \cap C$ or $\overline{M} \cap \overline{C}$ is finite.

This definition of maximal ω-r.e. set is due to Myhill [12]. Myhill hoped that maximal ω-r.e. sets existed, and that no maximal ω-r.e. set would have as its Turing degree, the complete ω-r.e. Turing degree. Friedberg [2] constructed a maximal ω-r.e. set, but Yates [22] then constructed a maximal ω-r.e. set with complete ω-r.e. Turing degree, ruining Myhill's proposed program.

As we mentioned in the introduction, we feel that the major importance of maximal ω-r.e. sets is their relationship to $\mathfrak{E}(\omega)$. Lachlan's [5] decision procedure for a certain natural set of sentences of $\mathfrak{L}$ over $\mathfrak{E}(\omega)$ uses the existence of maximal ω-r.e. sets, and their definability over $\mathfrak{E}(\omega)$. Hence a superior(5) definition of maximal α-r.e. set, we feel, should be definable over $\mathfrak{E}(\alpha)$. Since one may also want to study certain quotient lattices of $\mathfrak{E}(\alpha)$ modulo appropriate congruence relations, we feel that any reasonable definition of maximal α-r.e. set should yield a maximal element in one of these quotient lattices. With this in mind, we offer the following generalizations of finite.

For the rest of the paper, let $K = \{\alpha\} \cup \{\kappa: \kappa \text{ is an } \alpha\text{-cardinal}\}$. Let $x \in K$. We call $F \subseteq \alpha$ an *xB-finite set* if F is α-bounded and of order type $< x$; we call $F \subseteq \alpha$ an *xR-finite set* if F is an α-r.e. xB-finite set; and we call $F \subseteq \alpha$ an *xA-finite set* if F is an α-finite xB-finite set.

For the rest of the paper, let $J = \{xA, xB, xR: x \in K\}$. For $z \in J$, $z \notin \{\alpha A, \alpha R\}$ the z-finite sets form an ideal in $\mathfrak{E}(\alpha)$, hence if we define $C \sim_z D$ for $C, D \in \mathfrak{E}(\alpha)$, by $C \sim_z D$ if and only if $(C - D) \cup (D - C)$ is z-finite, then $\sim_z$ is a congruence relation. This is not the case, however, if $z \in \{\alpha A, \alpha R\}$ and $\alpha^* < \alpha$.

Let M be an α-r.e. set, and let $y, z \in J$. We call M a *yz-maximal α-r.e. set* if $\overline{M}$ is not y-finite, but for any α-r.e. set C, either $\overline{M} \cap C$ or $\overline{M} \cap \overline{C}$ is z-finite. The definitions of maximal α-r.e. set introduced by Kreisel and Sacks [3] are the yz-maximal α-r.e. sets for $y, z \in \{\omega B, \alpha A, \alpha B\}$.

From the preceding remarks, the only definitions of maximal α-r.e. set which we would consider reasonable are the zz-maximal α-r.e. sets for $z \in J$, $z \notin \{\alpha A, \alpha R\}$. For such z, we get maximal elements of the lattice $\mathfrak{E}(\alpha)/\sim_z$. Other definitions may be useful in that they may be definable over $\mathfrak{E}(\alpha)$ or their existence may imply the existence of maximal sets under some reasonable definition. We feel, however, that none of the other definitions should be thought of as defining maximal sets in the true sense of maximality for a lattice theoretic approach.

Of all the reasonable definitions mentioned in the above paragraph, we wish

(5) Again we mean in the lattice theoretic setting.

to single out one definition, $\alpha^*A\alpha^*A$-maximal, as being the most natural one at this time. This is because "F is α^*A-finite" is definable over $\mathcal{E}(\alpha)$, hence "M is $\alpha^*A\alpha^*A$-maximal" is definable over $\mathcal{E}(\alpha)$. It is easy to see that the predicates "$A \cup B = C$", "$A \cap B = C$", and "$A = \overline{B}$" are definable over $\mathcal{E}(\alpha)$. We use the same definitions as Lachlan [6]. Define "R is α-recursive" if and only if R has a complement in $\mathcal{E}(\alpha)$, and "F is finite" if and only if every subset of F is α-recursive.(6) "Finite" coincides with "α^*A-finite". Should other definitions of maximal α-r.e. set prove to be definable over $\mathcal{E}(\alpha)$ and yield maximal α-r.e. sets which are not $\alpha^*A\alpha^*A$-maximal, the situation would have to be reevaluated as to which definition is the best.

We conclude this section with some easy lemmas about maximal sets. These will be of use to us in subsequent sections.

Lemma 3.1. *Let $x, y \in K$, $X \in \{A, B, R\}$ and $v \in J$. Then*

(3.1) *If $x < y$ and M is a yXv-maximal α-r.e. set, then M is an xXv-maximal α-r.e. set.*

(3.2) *If $x < y$ and M is a vxX-maximal α-r.e. set, then M is a vyX-maximal α-r.e. set.*

(3.3) *If M is an xRv-maximal α-r.e. set, then M is an xAv-maximal α-r.e. set.*

(3.4) *If M is an xBv-maximal α-r.e. set, then M is an xRv-maximal α-r.e. set.*

(3.5) *If M is a vxA-maximal α-r.e. set, then M is a vxR-maximal α-r.e. set.*

(3.6) *If M is a vxR-maximal α-r.e. set, then M is a vxB-maximal α-r.e. set.*

Proof. Immediate from the definitions.

Lemma 3.2. *Let $\kappa \in K$, $X, Y \in \{A, B, R\}$. Assume that M is a $\kappa X \kappa Y$-maximal α-r.e. set. Then $\overline{M}$ is not an α-r.e. set.*

Proof. Assume to the contrary that $\overline{M}$ is an α-r.e. set. Since M is α-r.e., $\overline{M}$ must be α-recursive. Let $\overline{M}$ have order type β. Since $\overline{M}$ is α-recursive, if $\overline{M}$ is α-bounded, $\beta = \alpha$; hence in any case $\beta \geq \kappa$ since $\overline{M}$ is not κX-finite. Let $f: \beta \to \overline{M}$ be the α-recursive function enumerating $\overline{M}$ in order of magnitude. Let $C = \{y: (\exists x)(\exists u)(x = 2u < \beta \text{ and } f(x) = y)\}$. Since κ is an α-cardinal and $\beta \geq \kappa$, both $C \cap \overline{M}$ and $\overline{C} \cap \overline{M}$ must have order type $\geq \kappa$. So neither $\overline{M} \cap C$ nor $\overline{M} \cap \overline{C}$ can be κY-finite. But C is an α-r.e. set, so we have a contradiction which proves the lemma.

Lemma 3.3. *Let M be an $\alpha A \alpha R$-maximal α-r.e. set. Then either $\overline{M}$ is α-unbounded or M is an $\alpha^*A\alpha^*R$-maximal α-r.e. set.*

Proof. Assume to the contrary, that $\overline{M}$ is α-bounded, but that M is not an $\alpha^*A\alpha^*R$-maximal α-r.e. set. $\overline{M}$ cannot be α-recursive because of Lemma 3.2.

(6) These definitions, we believe, were first noticed by Lacombe.

Thus $\overline{M}$ is not $\alpha^* A$-finite. Hence there is an α-r.e. set C such that either $\overline{M} \cap C$ or $\overline{M} \cap \overline{C}$ is α-r.e., but neither $\overline{M} \cap C$ nor $\overline{M} \cap \overline{C}$ is $\alpha^* R$-finite. This can only happen if the intersection which is α-r.e. is an α-bounded set of order type $\geq \alpha^*$, and $\alpha^* < \alpha$.

Assume first that $\overline{M} \cap C$ is α-bounded, α-r.e., and of order type $\geq \alpha^*$. If $\overline{M} \cap C$ is α-recursive, let $D = \overline{M} \cap C$. Otherwise, by Lemma 2.3, choose D to be an α-finite subset of $\overline{M} \cap C$ of order type $\geq \alpha^*$. By Lemma 2.4, choose E to be an α-r.e. subset of D such that $D \cap \overline{E}$ is not α-recursive. Since D is α-recursive, $D \cap \overline{E}$ cannot be an α-r.e. set. Then $E \cup \overline{D}$ is α-r.e., so since M is $\alpha A \alpha R$-maximal, either $(E \cup \overline{D}) \cap \overline{M}$ or $\overline{(E \cup \overline{D})} \cap \overline{M}$ is an α-r.e. set.

If $(E \cup \overline{D}) \cap \overline{M}$ is α-r.e., then $((E \cup \overline{D}) \cap \overline{M}) \cup D$ is an α-r.e. set. But $((E \cup \overline{D}) \cap \overline{M}) \cup D = \overline{M}$ since $D \subseteq \overline{M}$, so $\overline{M}$ is α-r.e., contradicting Lemma 3.2. If $\overline{(E \cup \overline{D})} \cap \overline{M}$ is α-r.e., then $\overline{(E \cup \overline{D})} \cap \overline{M} = \overline{E} \cap \overline{\overline{D}} \cap \overline{M} = \overline{E} \cap D$ since $D \subseteq \overline{M}$, so $D \cap \overline{E}$ is α-r.e., contradicting the choice of E.

We must therefore conclude that $\overline{M} \cap \overline{C}$ is α-bounded, α-r.e., and of order type $\geq \alpha^*$. If $\overline{M} \cap \overline{C}$ is α-recursive, let $D = \overline{M} \cap \overline{C}$. Otherwise, by Lemma 2.3, choose D to be an α-finite subset of $\overline{M} \cap \overline{C}$ of order type $\geq \alpha^*$. By Lemma 2.4, choose E to be an α-r.e. subset of D such that $D \cap \overline{E}$ is not α-recursive. Since D is α-recursive $D \cap \overline{E}$ cannot be an α-r.e. set. Then $E \cup \overline{D}$ is α-r.e., so since M is $\alpha A \alpha R$-maximal, either $(E \cup \overline{D}) \cap \overline{M}$ or $\overline{(E \cup \overline{D})} \cap \overline{M}$ is an α-r.e. set. In either case, a contradiction is obtained as in the previous paragraph, proving the lemma.

Lemma 3.4. *Assume that there exists a $\kappa A \kappa B$-maximal α-r.e. set M_1, with $\kappa \in K$ and let $\gamma = \sup(\overline{M}_1)$. Then there exists a $\kappa A \kappa B$-maximal α-r.e. set M such that $\overline{M}$ has order type $\leq \kappa$, and $\overline{M} \subseteq \gamma$.*

Proof. Let M_1 be a $\kappa A \kappa B$-maximal α-r.e. set such that $\overline{M}_1$ has order type β and $\overline{M}_1 \subseteq [\gamma]$. We can assume that $\beta > \kappa$, else the lemma is proven.

Let λ be the least ordinal such that $\overline{M}_1 \cap [\lambda]$ has order type κ, and let $G = [\lambda, \alpha]$. Let $M = M_1 \cup G$. Then $\overline{M}$ has order type κ, so $\overline{M}$ is not κA-finite.

Let H be any α-r.e. set. Since M_1 is a $\kappa A \kappa B$-maximal α-r.e. set, either $\overline{M}_1 \cap H$ or $\overline{M}_1 \cap \overline{H}$ is κB-finite, i.e., has order type $< \kappa$ and is α-bounded. Since $\overline{M} \subseteq \overline{M}_1$, we have hereditarily that either $\overline{M} \cap H$ or $\overline{M} \cap \overline{H}$ is κB-finite; also $\overline{M} \subseteq [\gamma]$. This proves the lemma.

Lemma 3.5. *Let M_1 be a $\kappa A \kappa B$-maximal α-r.e. set, for $\kappa \in K$, such that $\overline{M}_1$ is α-bounded. Then there is a $\kappa A \kappa B$-maximal α-r.e. set $M \supseteq M_1$ and a $\gamma < \alpha$ such that $\overline{M} \subseteq [\gamma]$ and M is γ-regular.*

Proof. Let $\delta = \sup(\overline{M}_1)$. If M_1 is δ-regular, let $M = M_1$ and we are done. Assume, therefore, that M_1 is not δ-regular. Let γ be the least ordinal $< \delta$

such that M_1 is not $\gamma + 1$-regular. Let $M = M_1 \cup [\gamma, \alpha]$. Clearly $M_1 \subseteq M$ and $\overline{M} \subseteq [\gamma]$. If $\overline{M}$ were α-finite, then since $M_1 \cap [\gamma]$ is α-r.e., $M_1 \cap [\gamma]$ would be α-finite, so M_1 would be $\gamma + 1$-regular. Hence $\overline{M}$ cannot be κA-finite. Since $\overline{M} \subseteq \overline{M}_1$, the rest of the proof follows as in the last paragraph of the proof of Lemma 3.4.

Lemma 3.6. *Let M be a $\kappa A \kappa B$-maximal α-r.e. set such that $\overline{M}$ has order type β and $\overline{M}$ is α-unbounded. Assume that there is no $\kappa A \kappa B$-maximal α-r.e. set M_1 such that $\overline{M}_1$ is α-unbounded and has order type $< \beta$. Let $\xi < \beta$ be given. Then there is a $\nu \leq \alpha$ such that $\beta = \xi \cdot \nu$.*

Proof. Since $\xi < \beta$, there are $\nu \leq \alpha$ and $\mu < \xi$ such that $\beta = \xi \cdot \nu + \mu$. Assume that $\mu > 0$. Let λ be the least ordinal such that $\overline{M} \cap [\lambda]$ has order type $\xi \cdot \nu$. Then $\lambda < \alpha$. Let $M_1 = M \cup [\lambda]$. Then $\overline{M}_1$ is α-unbounded, so $\overline{M}_1$ is not κA-finite. Since $\overline{M}_1 \subseteq \overline{M}$, and M is $\kappa A \kappa B$-maximal, for any α-r.e. set H, either $H \cap \overline{M}$ or $\overline{H} \cap \overline{M}$ is κB-finite, so hereditarily, M_1 is $\kappa A \kappa B$-maximal. But $\overline{M}_1$ has order type $\mu < \xi < \beta$, contradiction. Thus $\mu = 0$, proving the lemma.

4. Nonexistence. Throughout this section, let $\kappa \in K$, $X \in \{A, B, R\}$, where K, A, B, and R are as defined in §3. We prove the nonexistence of $\kappa X \kappa X$-maximal α-r.e. sets for various α.

The first nonexistence theorem for maximal α-r.e. sets was proved by Sacks [16], who showed that there are no $\alpha B \alpha B$-maximal α-r.e. sets for $\alpha = \aleph_1^L$. In [9], Lerman and Simpson proved the nonexistence of several types of maximal α-r.e. sets for all $\alpha \geq \aleph_1^L$. Our main lemmas in this section are stronger versions of those in [9], the proofs here being somewhat more complicated.

For much of the work in this section, it suffices to consider only $\kappa A \kappa B$-maximal α-r.e. sets. For the sake of notational convenience, we call such sets κ-maximal throughout this section.

Our first theorem gives us one of the two conditions on α necessary for the existence of a κ-maximal α-r.e. set.

Theorem 4.1. *If there exists a κ-maximal α-r.e. set, then* $\mathrm{s2cf}(\alpha) \leq \kappa$.

Proof. Assume that M is a κ-maximal α-r.e. set. By Lemma 3.4, we can assume that $\beta \leq \kappa$, where β is the order type of $\overline{M}$. By Lemma 3.2, $\overline{M}$ is not α-r.e., hence $\overline{M}$ is not α-recursive. Hence by Lemma 2.10, $\mathrm{s2cf}(\alpha) \leq \beta \leq \kappa$.

Much of the remainder of this section is devoted to proving the following theorem.

Theorem 4.2. *Assume either that there exists a κ-maximal α-r.e. set for some $\kappa \leq \alpha^*$, or that there exists an $\alpha B \alpha B$-maximal α-r.e. set. Then* $\mathrm{s3p}(\alpha) = \omega$.

Proof. Let γ be the least ordinal $\mu \leq \alpha$ such that there exists a κ-maximal α-r.e. set P with $\overline{P} \subseteq \mu$ if $\kappa \leq \alpha^*$, and let $\gamma = \alpha$ otherwise. Let β be the least ordinal $\nu \leq \alpha$ such that there exists a κ-maximal α-r.e. set P with $\overline{P} \subseteq \gamma$ and such that $\overline{P}$ has order type ν if $\kappa \leq \alpha^*$, and let β be the least order type of the complement of a $\kappa B \kappa B$-maximal α-r.e. set if $\alpha^* < \alpha = \kappa$. Let M be a κ-maximal α-r.e. set such that $\overline{M} \subseteq \gamma$ and $\overline{M}$ has order type β if $\kappa \leq \alpha^*$, and let M be an $\alpha B \alpha B$-maximal α-r.e. set such that $\overline{M} \subseteq \gamma$ and $\overline{M}$ has order type β if $\alpha^* < \alpha = \kappa$. Let $\delta =$ s2cf(α), $\pi_2 =$ ts2p(α), and $\pi_3 =$ s3p(α).

Let ϕ be a one-one α-recursive function enumerating M. For each $\sigma < \alpha$, let $M_\sigma = \{y: (\exists x)(x < \sigma \text{ and } \phi(x) = y)\}$. For each $\sigma < \alpha$, let $\{m_i^\sigma: i < \alpha\}$ be the elements of $\overline{M}_\sigma$ in order of magnitude, and let $\{m_i: i < \beta\}$ be the elements of $\overline{M}$ in order of magnitude.

If $\pi_3 = \omega$, we are done. Hence we assume that $\pi_3 > \omega$. Under this assumption, we show that $\pi_3 = \omega$, a contradiction which proves the lemma.

We now outline the rest of the proof. If $\overline{M}$ is α-bounded, we show that $\pi_2 \leq \beta$. If $\overline{M}$ is α-unbounded, we show that $\pi_2 \leq \delta$. Using these facts, we show that s3cf(π_2) $= \omega$ and that s3p(ν) $= \omega$ uniformly for $\nu < \pi_2$. The theorem then follows from Lemma 2.22 and Lemma 2.19.

Lemma 4.3. *If $\overline{M}$ is α-bounded, then $\pi_2 \leq \beta$.*

Proof. By Lemma 3.2, M is not α-recursive. Since $\gamma < \alpha$, by Lemma 2.6, $\alpha^* \leq \beta \leq \gamma < \alpha$. By Lemma 2.12, $\pi_2 =$ ts2p(α) $\leq \alpha^*$. Hence $\pi_2 \leq \alpha^* \leq \beta$.

Lemma 4.4. *If $\overline{M}$ is α-unbounded, then $\pi_2 \leq \delta$.*

Proof. Let $h': \alpha \times \delta \to \alpha$ generate a strictly increasing tame S_2 cofinality function $h: \delta \to \alpha$. By Lemma 2.9, h exists, and we can assume that h' witnesses the tameness of h. If $\delta = \alpha$, we are done. We therefore assume that $\delta < \alpha$. We will partition α using an α-recursive sequence of α-finite sets, $\{A_i: i < \delta\}$. The A_i will be constructed by induction on the set of stages, $\{\sigma: \sigma < \alpha\}$.

Stage 0. For all $i < \delta$, let $A_i^0 = \emptyset$.

Stage $\sigma > 0$. Let j be the least $i < \delta$ such that $h'(\sigma, i) \neq \lim_{\tau \to \sigma} h'(\tau, i)$ if such an i exists. If no such i exists, set $A_i^\sigma = \bigcup_{\tau < \sigma} A_i^\tau$ for all $i < \delta$ and go to the next stage. Assume that j exists. Let $D^\sigma = \{x \leq \sigma: x \notin \bigcup_{i < \delta} \bigcup_{\tau < \sigma} A_i^\tau\}$. Define $A_i^\sigma = \bigcup_{\tau < \sigma} A_i^\tau$ for all $i < \delta$ such that $i \neq j$, and $A_j^\sigma = \bigcup_{\tau < \sigma} A_j^\sigma \cup D^\sigma$. Then go to the next stage.

For each $i < \delta$, set $A_i = \bigcup_{\sigma < \alpha} A_i^\sigma$. Since the construction is α-effective, each A_i is an α-r.e. set.

We next prove the following assertions about the construction.

(4.1) If $i < \delta$, $j < \delta$, and $i \neq j$, then $A_i \cap A_j = \emptyset$.

(4.2) For all $x < \alpha$, there is an $i < \delta$ such that $x \in A_i$.

(4.3) For all $i < \delta$, there is a $\mu < \alpha$ such that $\sup(\bigcup_{j \leq i} A_j) \leq \mu$.

Since $(\bigcup_{i<\delta} A_i^\sigma) - (\bigcup_{i<\delta} \bigcup_{\tau<\sigma} A_i^\tau) = D^\sigma$, and since $D^\sigma \cap (\bigcup_{i<\delta} \bigcup_{\tau<\sigma} A_i^\tau) = \emptyset$, (4.1) follows immediately.

Let $x < \alpha$ be given. If $x \notin \bigcup_{i<\delta} A_i$, then for all $i < \delta$, $h(i) = \lim_{\sigma \to \alpha} h'(\sigma, i) = h'(x, i)$. Hence h must be α-recursive, contradicting the admissibility of α since $\delta < \alpha$. Thus $x \in \bigcup_{i<\delta} A_i$, proving (4.2).

Since h' generates h as a tame S_2 function, for each $i < \delta$ there is a $\mu < \alpha$ such that $h'(\sigma, j) = h'(\mu, j)$ for all $\sigma \geq \mu$ and $j \leq i$. From the construction, we must therefore have $\sup(\bigcup_{j \leq i} A_j) = \sup(\bigcup_{j \leq i} A_j^\mu) \leq \mu$, proving (4.3).

We now define $I \subseteq [\delta]$ by $i \in I \Leftrightarrow (\exists x)(x \in A_i \cap \overline{M}$ and $(y)(j)(j < i$ and $y \in A_j \cap \overline{M} \Rightarrow x > y))$. We prove the following assertions about I.

(4.4) I is cofinal with δ.

(4.5) I is a tame S_2 set.

(4.6) I is not α-finite.

Let $i < \delta$ be given. By (4.3), there is a $\mu < \alpha$ such that $\sup(\bigcup_{j \leq i} A_j) \leq \mu$. Let j be the least $n < \delta$ such that $n > i$ and such that there exists an $x \in \overline{M} \cap A_n \cap [\mu, \alpha]$. Since $\overline{M}$ is α-unbounded, such a j must exist by (4.2). But by definition of I, $j \in I$, proving (4.4).

Define I': $\alpha \times \delta \to \{0, 1\}$ as follows. $I'(\sigma, i) = 1$ if there is an x in $\overline{M}_\sigma \cap A_i^\sigma$ such that for all $y \in \overline{M}_\sigma \cap (\bigcup_{j<i} A_j^\sigma)$, $x > y$; and $I'(\sigma, i) = 0$ otherwise. I' is clearly an α-recursive function. Let $j < \delta$ be given. Assume first that $I(j) = 1$, i.e., $j \in I$. Let $\nu = \sup((\bigcup_{n<j} A_n) \cap \overline{M})$. By (4.3), $\nu < \alpha$. Since $j \in I$, there is a least $x \in \overline{M} \cap A_j$ such that $x > \nu$. By (4.3), let $\sup(\bigcup_{n \leq j} A_n) = \sigma < \alpha$. Then for each $n \leq j$, $A_n = A_n^\sigma$, so $(\bigcup_{n \leq j} A_n) \cap [\nu, x]$ is an α-finite subset of M, hence there is a stage $\tau \geq \sigma$ such that $(\bigcup_{n \leq j} A_n^\tau) \cap [\nu, x] \subseteq M_\tau$ and $x \in A_j^\tau$. We now note that for all $\lambda \geq \tau$, $I'(\lambda, j) = 1$, so $I(j) = 1 = \lim_{\lambda \to \alpha} I'(\lambda, j)$. Now assume that $I(j) = 0$, i.e., $j \notin I$. Let $\nu = \sup((\bigcup_{n<j} A_n) \cap \overline{M})$, let $x = \sup(A_j)$, and let $\sigma = \sup(\bigcup_{n \leq j} A_n)$. Again by (4.3), $\nu < \alpha$, $x < \alpha$, and $\sigma < \alpha$. Since $A_j \cap [\nu, x] \subseteq M$, there is a stage $\tau \geq \sigma$ such that $A_j \cap [\nu, x] \subseteq M_\tau$. We note that for all $\lambda \geq \tau$, $I'(\lambda, j) = 0$, so $I(j) = 0 = \lim_{\lambda \to \alpha} I'(\lambda, j)$. Hence I' generates I as an S_2 function. Since $\delta = \mathrm{s2cf}(\alpha)$, I is tame by Lemma 2.8. Thus (4.5) is proved.

Assume that I is α-finite. Then there is a one-one α-finite function θ, enumerating I in order of magnitude. Define $I_1 \subseteq I$ by $x \in I_1 \Leftrightarrow (\exists y)(\exists z)$ $(y = 2 \cdot z$ and $\theta(y) = x)$. Let $I_2 = I - I_1$. Then I_2 and I_1 are disjoint α-finite subsets of I, so since δ is a limit ordinal by Lemma 2.7, (4.4) tells us that I_1 and I_2 are both cofinal with δ. Let $B_1 = \bigcup\{A_i: i \in I_1\}$ and let $B_2 = \bigcup\{A_i: i \in I_2\}$. By (4.1), B_1 and B_2 are disjoint α-r.e. sets. Let $\nu < \alpha$ be given. By (4.2), there is an $i < \delta$ and a $z > \nu$ such that $z \in A_i$. Fix such an i and $z \in A_i$. Let $\mu = \sup((\bigcup_{j \le i} A_j) \cap \overline{M})$. By (4.3), $\mu < \alpha$. Since I_1 and I_2 are cofinal with δ and subsets of I, there are $m \in I_1$ and $n \in I_2$ such that $i < m < \delta$, $i < n < \delta$, and x and y such that $x > \mu$, $y > \mu$, $x \in A_m \cap \overline{M}$, and $y \in A_n \cap \overline{M}$. Note that $x \in B_1$ and $y \in B_2$. Since $\nu < z < \mu < x$ and $\nu < z < \mu < y$, both $\overline{M} \cap B_1$ and $\overline{M} \cap B_2$ must be α-unbounded. Since $\overline{M} \cap \overline{B}_1 \supseteq \overline{M} \cap B_2$, neither $\overline{M} \cap B_1$, nor $\overline{M} \cap \overline{B}_1$ can be κB-finite, so M cannot be κ-maximal if $\kappa \le \alpha^*$, and M cannot be $\alpha B \alpha B$-maximal if $\alpha^* < \alpha = \kappa$. This contradiction proves (4.6).

We now note that by (4.5), (4.6), and Lemma 2.7, the hypothesis of Lemma 2.13 is satisfied. Hence by Lemma 2.13, $\pi_2 = \mathrm{ts2p}(\alpha) \le \delta$, proving the lemma.

We return to the proof of Theorem 4.2. For each ξ such that $\omega \le \xi < \pi_2$, we construct an α-recursive sequence $\{C_i^\xi: i < \xi\}$ of α-r.e. sets. The construction proceeds by induction on the set of stages $\{\sigma: \sigma < \alpha\}$. Fix ξ.

Stage 0. For all $i < \xi$ let $C_{i,0}^\xi = \emptyset$.

Stage $\sigma > 0$. For all $i < \xi$, let $C_{i,\sigma}^\xi = (\bigcup_{\lambda < \sigma} C_{i,\lambda}^\xi) \cup \{x: x < \sigma \text{ and } x < y \text{ and } (\exists j)(\exists \lambda)(x = m_j^\sigma \text{ and } j = \xi \cdot \lambda + i)\}$.

This completes the construction. For each $i < \xi$, let $C_i^\xi = \bigcup_{\sigma < \alpha} C_{i,\sigma}^\xi$. Clearly for all $i < \xi$, C_i^ξ is an α-r.e. set.

We now define an α-recursive function $g': \alpha \times \alpha \times \pi_2 \times \pi_2 \to y$. For each ξ such that $\omega \le \xi < \pi_2$, $\lambda \sigma \tau x g'(\sigma, \tau, x, \xi)$ will generate an S_3 cofinality function. If $\xi < \omega$ or if $\lambda \ge \xi$, $g'(\sigma, \tau, \lambda, \xi) = 0$. If $\omega \le \xi < \pi_2$ and $\lambda < \xi$, we define g' by cases.

Case 1. $y = \alpha$. Define $g'(\sigma, \tau, \lambda, \xi) = \sup(\overline{C}_{\lambda,\tau}^\xi \cap \overline{M}_\tau \cap [\sigma])$.

Case 2. $y < \alpha$. Define $g'(\sigma, \tau, \lambda, \xi) =$ the order type of $\overline{C}_{\lambda,\tau}^\xi \cap \overline{M}_\tau \cap [y]$.

Lemma 4.5. *Let ξ and x be given such that $\omega \le \xi < \pi_2$ and $x \in \overline{M}$. Let $x = m_j$. Then*

(4.7) *there are only finitely many $r < \xi$ such that $x \in C_r^\xi$; and*

(4.8) $x = \lim_{\sigma \to \alpha} m_j^\sigma$.

Proof. We note that if $x = m_i^\sigma = m_n^\tau$ and $\tau \ge \sigma$, then since $\overline{M}_\tau \subseteq \overline{M}_\sigma$, we must have $n \le i$. Thus $\{r < \xi: x \in C_r^\xi\}$ must be finite, else we would be able to define an infinite decreasing sequence of ordinals. This proves (4.7).

Since $\overline{M}$ has order type β, $j < \beta$. We prove (4.8) by induction on $\{n: n < \beta\}$. By induction, we may assume that $m_i = \lim_{\sigma \to \alpha} m_i^\sigma$ for all $i < j$. Let $\nu = \sup(\{m_i: i < j\})$. Then $\nu < x$, and for each $y < x$, there is a stage σ such that $\{z: z < y \text{ and } z \in \overline{M}_\sigma\}$ has order type $< j$; for if $y < \nu$, there is an $x_0 \in \overline{M}$

such that $y \leq x_0 < \nu$, and since $x_0 = \lim_{\sigma \to \alpha} m_i^\sigma$ for some $i < j$, such a σ must exist. For all $y < \nu$, let $\theta(y)$ be the least σ such that $[y] \cap \overline{M}_\sigma$ has order type $< j$. Then $\theta: \nu \to \alpha$ is an α-recursive function with α-finite domain, so $\sup(\theta([\nu])) < \alpha$ by the admissibility of α. Let $\lambda = \sup(\theta([\nu]))$. Since $[\nu, x] \subseteq M$, there is a stage ρ such that $[\nu, x] \subseteq M_\rho$ and $\lambda \leq \rho < \alpha$. We note that for all $\tau \geq \rho$, $[x] \cap \overline{M}_\tau$ has order type j, so $x = m_j^\rho = \lim_{\tau \to \alpha} m_j^\tau$, proving (4.8).

Lemma 4.6. *For each ξ such that $\omega \leq \xi < \pi_2$, $\lambda\sigma\tau x g'(\sigma, \tau, x, \xi)$ generates an S_3 function $\lambda x g(x, \xi)$ such that*

(4.9) *if $\gamma = \alpha$ and $x < \xi$, then $g(x, \xi) = \sup(\overline{M} \cap \overline{C}_x^\xi) < \alpha$; and*

(4.10) *if $\gamma < \alpha$ and $x < \xi$, then $\beta = \alpha^*$ and $g(x, \xi)$ is the order type of $\overline{M} \cap \overline{C}_x^\xi$, which is $< \beta$.*

Proof. Fix ξ such that $\omega \leq \xi < \pi_2$, and fix $x < \xi$. We proceed by cases.

Case 1. $\gamma = \alpha$. By Lemma 4.4 and Lemma 2.10, $\pi_2 \leq \delta \leq \beta$, so $\xi < \beta$. Since every $\kappa B \kappa B$-maximal α-r.e. set P with $\overline{P}$ α-unbounded is a $\kappa A \kappa B$-maximal α-r.e. set, by Lemma 3.6, there is a $\nu \leq \alpha$ such that $\beta = \xi \cdot \nu$. By (4.8) of Lemma 4.5 and the definition of C_x^ξ, if $\mu < \nu$, then $m_{\xi \cdot \mu + x} \in C_x^\xi$. Thus $\overline{M} \cap C_x^\xi$ is α-unbounded since $\overline{M}$ is α-unbounded and $\beta = \xi \cdot \nu$. Since C_x^ξ is α-r.e. and M is either κ-maximal or $\kappa B \kappa B$-maximal, $\overline{M} \cap \overline{C}_x^\xi$ must be α-bounded.

Since C_x^ξ and M are α-r.e. sets, if $\tau < \lambda < \alpha$ and $\sigma < \alpha$, then $\overline{C}_{x,\tau}^\xi \cap \overline{M}_\tau \cap [\sigma] \supseteq \overline{C}_{x,\lambda}^\xi \cap \overline{M}_\lambda \cap [\sigma]$. Hence $\lambda\tau g'(\sigma, \tau, x, \xi)$ is nonincreasing for $\tau \geq \sigma$. Thus $\lim_{\tau \to \alpha} g'(\sigma, \tau, x, \xi)$ exists, else we could define an infinite decreasing sequence of ordinals. Let $y = \sup(\overline{C}_x^\xi \cap \overline{M})$. We have shown in the preceding paragraph that $y < \alpha$. If $\sigma > y$, then since $[y, \sigma] \subseteq M \cup C_x^\xi$, $[y, \sigma]$ is α-finite, and $M \cup C_x^\xi$ is an α-r.e. set, there is a $\tau < \alpha$ such that $[y, \sigma] \subseteq C_{x,\lambda}^\xi \cup M_\lambda$, for all λ such that $\tau \leq \lambda < \alpha$. Hence if $\sigma > y$, then $\lim_{\tau \to \alpha} g'(\sigma, \tau, x, \xi) \leq y$. But clearly $\lim_{\tau \to \alpha} g'(\sigma, \tau, x, \xi) \geq y$. Thus $\lim_{\sigma \to \alpha} \lim_{\tau \to \alpha} g'(\sigma, \tau, x, \xi) = y$. Thus $\lambda x g(x, \xi)$ is an S_3 function on domain $[\xi]$ as defined in (4.9).

Case 2. $\gamma < \alpha$. Then by the hypothesis of the theorem, $\kappa \leq \alpha^*$. By Lemma 3.4, $\beta \leq \alpha^*$. By Lemma 3.2 and Lemma 2.6, $\beta \geq \alpha^*$. Hence $\beta = \alpha^*$. By Lemma 2.1, $\beta = \xi \cdot \alpha^*$. By (4.8) of Lemma 4.5 and the definition of C_x^ξ, if $\mu < \alpha^*$ then $m_{\xi \cdot \mu + x} \in C_x^\xi$. Thus $C_x^\xi \cap \overline{M}$ must have order type α^*. Since $\kappa \leq \alpha^*$, $C_x^\xi \cap \overline{M}$ cannot be κB-finite, so since C_x^ξ is α-r.e. and M is κ-maximal, $\overline{C}_x^\xi \cap \overline{M}$ must have order type $< \alpha^*$. Furthermore, since $C_x^\xi \cup M$ is α-r.e., by Lemma 2.6, $\overline{C}_x^\xi \cap \overline{M} = \overline{C_x^\xi \cup M}$ must be α-finite. Consequently, $(C_x^\xi \cup M) \cap [y]$ is α-finite.

Since $(C_x^\xi \cup M) \cap [y]$ is α-finite, there is a stage τ such that for all $\lambda \geq \tau$, $(C_{x,\lambda}^\xi \cup M_\lambda) \cap [y] = (C_x^\xi \cap M) \cap [y]$. Fix such a τ. Then for all $\lambda \geq \tau$,

$\overline{C}^{\xi}_{x,\lambda} \cap \overline{M}_\lambda \cap [\gamma] = \overline{C}^{\xi}_x \cap \overline{M} \cap [\gamma]$. By definition, $\lim_{\sigma \to \alpha} \lim_{\tau \to \alpha} g'(\sigma, \tau, x, \xi) = \lim_{\tau \to \alpha} g'(0, \tau, x, \xi) = \lim_{\tau \to \alpha}$ (the order type of $\overline{C}^{\xi}_{x,\tau} \cap \overline{M}_\tau \cap [\gamma]$) = the order type of $\overline{C}^{\xi}_x \cap \overline{M} \cap [\gamma]$ = the order type of $\overline{C}^{\xi}_x \cap \overline{M}$. Hence $\lambda x g'(x, \xi)$ is an S_3 function on domain $[\xi]$ satisfying (4.10).

We note that if $x \geq \xi$, $g'(\sigma, \tau, x, \xi) = 0$ for all $\sigma < \alpha$ and $\tau < \alpha$, so g' generates an S_3 function on all of its domain. This concludes the proof of the lemma.

Lemma 4.7. *For each ξ such that $\omega \leq \xi < \pi_2$,*
(4.11) *if $\gamma = \alpha$, then $\lambda x g(x, \xi)\restriction_\xi : \xi \to \alpha$ is an S_3 cofinality function*; *and*
(4.12) *if $\gamma < \alpha$, then $\lambda x g(x, \xi)\restriction_\xi : \xi \to \alpha^*$ is an S_3 cofinality function.*

Proof. By Lemma 4.6, $\lambda x g(x, \xi)\restriction_\xi$ is an S_3 function. Fix ξ such that $\omega \leq \xi < \pi_2$. We proceed by cases.

Case 1. $\gamma = \alpha$. We must show that $g([\xi], \xi)$ is cofinal with α. Let $\lambda < \alpha$ be given. Since $\overline{M}$ is cofinal with α, there is an $x \in \overline{M}$ such that $x > \lambda$. Since $\xi \geq \omega$, by (4.7) there is an $i < \xi$ such that $x \notin C^{\xi}_i$. Hence by (4.9), $g(i, \xi) > \lambda$, proving (4.11).

Case 2. $\gamma < \alpha$. We must show that $g([\xi], \xi)$ is cofinal with α^*. Assume for the sake of contradiction that $\sup(g([\xi], \xi)) = \mu < \alpha^*$. Since $\kappa \leq \alpha^*$ and $\gamma < \alpha$, by Lemma 3.2, Lemma 2.6, and Lemma 3.4, $\beta = \alpha^* < \alpha$. For each $i < \xi$, $g(i, \xi) < \beta = \alpha^*$ by (4.10), and $g(i, \xi)$ is the order type of $\overline{M} \cap \overline{C}^{\xi}_i$. By Lemma 2.6, $\overline{M} \cap \overline{C}^{\xi}_i = M \cup C^{\xi}_i$ is α-finite, since M and C^{ξ}_i are α-r.e. sets. Hence $(M \cup C^{\xi}_i) \cap [\gamma]$ is α-finite. Thus there is a stage σ such that for all $\tau \geq \sigma$, $(M_\tau \cup C^{\xi}_{i,\tau}) \cap [\gamma] = (M \cup C^{\xi}_i) \cap [\gamma]$. Define $\theta: \xi \to \alpha$ by $\theta(i)$ is the least stage σ such that $\overline{M}_\sigma \cap \overline{C}^{\xi}_{i,\sigma}$ has order type $< \mu$. θ is α-recursive, and since $\xi < \pi_2 \leq \alpha$ and α is admissible, $\theta([\xi])$ must be α-finite. Let $\lambda = \sup(\theta([\xi]))$. Then for all $\tau > \lambda$ and $i < \xi$, $\overline{M}_\tau \cap \overline{C}^{\xi}_{i,\tau}$ has order type $< \mu$. Since $\xi \geq \omega$, by (4.7), for each $x \in \overline{M}$ there is an $i < \xi$ such that $x \notin C^{\xi}_i$. By choice of λ, we must therefore conclude that $\overline{M} \subseteq \bigcup\{\overline{C}^{\xi}_{i,\lambda} \cap \overline{M}_\lambda : i < \xi\}$. Since $\xi < \pi_2 \leq \alpha^*$ by Lemma 2.12, and since $\mu < \alpha^*$, $\bigcup\{\overline{C}^{\xi}_{i,\lambda} \cap \overline{M}_\lambda : i < \xi\}$ has α-cardinality $< \alpha^*$ (recall that α^* is an α-cardinal, so there is a regular α-cardinal $\eta \leq \alpha^*$ such that $\eta > \mu$ and $\eta > \xi$ (namely, η = least α-cardinal $\geq \max\{\mu, \xi\}$); if the above union had α-cardinality $\geq \alpha^*$, we would contradict the regularity of η), hence has order type $< \alpha^*$. Thus $\overline{M} \subseteq \bigcup\{\overline{C}^{\xi}_{i,\lambda} \cap \overline{M}_\lambda : i < \xi\}$ so $\overline{M}$ has order type $< \alpha^*$, contradicting the fact that $\alpha^* = \beta$ = order type of $\overline{M}$, and proving (4.12).

Lemma 4.8. $\mathrm{s3cf}(\pi_2) = \omega$.

Proof. If $\gamma = \alpha$, choosing $\xi = \omega$ shows $\mathrm{s3cf}(\alpha) = \omega$. If $\gamma < \alpha$, then by (4.12), $\mathrm{s3cf}(\alpha^*) = \omega$. By Lemma 2.20, $\mathrm{s3cf}(\alpha) = \omega$. By Lemma 2.21, $\mathrm{s3cf}(\pi_2) \leq \omega$, so since π_2 is a limit ordinal, s3cf$(\pi_2) = \omega$, proving the lemma.

We return to the proof of Theorem 4.2. Define an α-recursive function $h': \alpha \times \alpha \times \omega \times \pi_2 \to \pi_2$ as follows. If $\xi < \omega$, $h'(\sigma, \tau, i, \xi) = i$ if $i < \xi$, and $h'(\sigma, \tau, i, \xi) = 0$ if $i \geq \xi$. If $\omega \leq \xi < \pi_2$, we define the ordering $<^{\xi}_{\sigma,\tau}$ of $[\xi]$ by $i <^{\xi}_{\sigma,\tau} j$ if $g'(\sigma, \tau, i, \xi) < g'(\sigma, \tau, j, \xi)$, or $g'(\sigma, \tau, i, \xi) = g'(\sigma, \tau, j, \xi)$ and $i < j$. Then $\lambda x h'(\sigma, \tau, x, \xi)$ enumerates the first ω elements of $[\xi]$ in order of magnitude according to the ordering prescribed by $<^{\xi}_{\sigma,\tau}$.

Similarly, for $\omega \leq \xi < \pi_2$, we define the ordering $<^{\xi}$ of $[\xi]$ by $i <^{\xi} j$ if $g(i, \xi) < g(j, \xi)$ or $g(i, \xi) = g(j, \xi)$ and $i < j$, where g is given by Lemma 4.6. $h: \omega \times \pi_2 \to \pi_2$ is defined by $h(i, \xi) = i$ if $i < \xi < \omega$, $h(i, \xi) = 0$ if $\xi \leq i < \omega$, and $\lambda x h(x, \xi)$ enumerates the first ω elements of $[\xi]$ in order of magnitude according to the ordering prescribed by $<^{\xi}$, if $\xi \geq \omega$.

Lemma 4.9. *For all $\xi < \pi_2$, $\lambda\sigma\tau x h'(\sigma, \tau, x, \xi)$ generates $\lambda x h(x, \xi)$ as an S_3 function.*

Proof. The lemma is clear if $\xi < \omega$. Fix ξ such that $\omega \leq \xi < \pi_2$. Define the ordering $<^{\xi}_{\sigma}$ of $[\xi]$ by $i <^{\xi}_{\sigma} j$ if $\lim_{\tau \to \alpha} g'(\sigma, \tau, i, \xi) < \lim_{\tau \to \alpha} g'(\sigma, \tau, j, \xi)$, or $\lim_{\tau \to \alpha} g'(\sigma, \tau, i, \xi) = \lim_{\tau \to \alpha} g'(\sigma, \tau, j, \xi)$ and $i < j$. Since g' generates an S_3 function, $\lim_{\tau \to \alpha} g'(\sigma, \tau, x, \xi)$ exists for all $\sigma < \alpha$ and $x < \xi$, so $<^{\xi}_{\sigma}$ is well defined for all $\sigma < \alpha$. Since $\lambda\tau g'(\sigma, \tau, x, \xi)$ is a nonincreasing function, for each $\sigma < \alpha$, if $x_0 <^{\xi}_{\sigma} x_1 <^{\xi}_{\sigma} \cdots <^{\xi}_{\sigma} x_n$ are the first $n+1$ elements of $[\xi]$ under $<^{\xi}_{\sigma}$, and if λ is chosen such that for all $\tau \geq \lambda$ and $i \leq n$, $g'(\sigma, \tau, x_i, \xi) = \lim_{\tau \to \alpha} g'(\sigma, \tau, x_i, \xi)$, then $x_0 <^{\xi}_{\sigma,\tau} x_1 <^{\xi}_{\sigma,\tau} \cdots <^{\xi}_{\sigma,\tau} x_n$ are the first $n+1$ elements of $[\xi]$ under $<^{\xi}_{\sigma,\tau}$ for all $\tau \geq \lambda$. Hence $\lim_{\tau \to \alpha} h'(\sigma, \tau, i, \xi)$ exists for all $\sigma < \alpha$ and $i < \omega$, and yields the first ω many elements of $[\xi]$ under $<^{\xi}_{\sigma}$ in the correct order. If $\gamma < \alpha$, then the definition of g' did not depend on σ, so $<^{\xi}_{\sigma}$ and $<^{\xi}$ coincide, for all $\xi < \alpha$, and $\lim_{\sigma \to \alpha} \lim_{\tau \to \alpha} h'(\sigma, \tau, i, \xi) = \lim_{\tau \to \alpha} h'(0, \tau, i, \xi)$, so h' generates h as an S_3 function.

Assume that $\gamma = \alpha$. We proceed by induction on $\{n: n < \omega\}$. Assume by induction that for all $j < n$, there is a stage s_j such that for all $\sigma \geq s_j$, $\lim_{\tau \to \alpha} h'(\sigma, \tau, j, \xi) = h(j, \xi)$. Fix such stages $\{s_j: j < n\}$.

Let t_0 be the least stage $\mu < \alpha$ such that $\mu \geq \max(\{s_j: j < n\})$ and for all $j \leq n$ and $\sigma \geq \mu$, $\lim_{\tau \to \alpha} g'(\sigma, \tau, h(j, \xi), \xi) = g(h(j, \xi), \xi)$. t_0 must exist since n is finite and g' generates g as an S_3 function. Let $\lambda_0 = \max(\{g(h(j, \xi), \xi): j \leq n\})$. Let x be the least element of $\overline{M}$ such that $x > \lambda_0$. Since $\overline{M}$ is α-unbounded, x must exist. By (4.7) and the definition of g', $L = \{z: (\exists \sigma > x)(\exists \tau \geq \sigma)(g'(\sigma, \tau, z, \xi) < x)\}$ must be finite. Choose $s_n \geq t_0$ such that for all $z \in L$ and all $\sigma > s_n$, $\lim_{\tau \to \alpha} g'(\sigma, \tau, z, \xi) = g(z, \xi)$. We note that $\{h(j, \xi): j < n\} \subseteq L$.

Fix σ such that $s_n < \sigma < \alpha$. Let t_1 be the least t such that for all $\tau \geq t$ and all $z \in L$, $g'(\sigma, \tau, z, \xi) = g(z, \xi)$. t_1 exists since L is finite. Then for all $\tau \geq t_1$, $h'(\sigma, \tau, n, \xi) = h(n, \xi)$. Thus the induction hypothesis is verified for $j = n$, and the lemma is proved.

Lemma 4.10. *Assume that $\xi < \pi_2$. Then for each $x < \xi$, there is an $i < \omega$ such that $h(i, \xi) = x$.*

Proof. The lemma is clear if $\xi < \omega$. Fix ξ such that $\omega \leq \xi < \pi_2$, and fix $x < \xi$. Let $g(x, \xi) = \nu$. Assume first that $\gamma = \alpha$. Let $z \in \overline{M}$ be such that $z > \nu$. Then by (4.7) and (4.9), $g(y, \xi) \leq \nu$ for only finitely many $y < \xi$, since $z \in \overline{M} \cap \overline{C}^{\xi}_y$ for all but finitely many $y < \xi$. Hence $h(i, \xi) = x$ for some $i < \omega$ by the definition of h.

Assume now that $\gamma < \alpha$. As noted in the proof of Lemma 4.9, for all $y < \xi$, $g(y, \xi) = \lim_{\tau \to \alpha} g'(0, \tau, y, \xi)$. If there are only finitely many $y < \xi$ such that $g(y, \xi) \leq \nu$, we are done by the definition of h. So we may assume that there are infinitely many $y < \xi$ such that $g(y, \xi) \leq \nu$, and derive a contradiction to prove the lemma. Recall that $\lambda \tau g'(0, \tau, y, \xi)$ is nonincreasing for all $y < \xi$.

Let $S = \{\tau\colon (\exists y < \xi)(g'(0, \tau, y, \xi) \leq \nu \text{ and } (\sigma)(\sigma < \tau \Rightarrow g'(0, \sigma, y, \xi) > \nu))\}$. If S were α-unbounded, then defining $p\colon S \to \xi$ by $p(\tau) = y$ if y is the least $z < \xi$ such that $g'(0, \tau, z, \xi) \leq \nu$ and $g'(0, \sigma, z, \xi) > \nu$ for all $\sigma < \tau$, since S is α-recursive hence of order type α by the admissibility of α, p would be a one-one α-recursive function. Hence $\alpha^* \leq \xi < \pi_2 \leq \alpha^*$ by Lemma 2.12, yielding a contradiction. Hence S must be α-bounded. Let $\lambda = \sup(S)$. Let $J = \{y\colon y < \xi$ and $g'(0, \lambda, y, \xi) \leq \nu\}$. Since $g(y, \xi) \leq \nu$ for infinitely many $y < \xi$ and by choice of λ, J must be infinite, but clearly J is α-finite. If $x \in \overline{M}$ then $x < y$, and by (4.7), $x \in \overline{C}^{\xi}_{y,\lambda}$ for all but finitely many $y \in J$. Hence $\overline{M} \subseteq \bigcup_{j \in J}(\overline{M}_\lambda \cap \overline{C}^{\xi}_{j,\lambda} \cap [y])$. But $\operatorname{card}(J) < \alpha^*$ and for each $j \in J$, $\operatorname{card}(\overline{M}_\lambda \cap \overline{C}^{\xi}_{j,\lambda} \cap [y]) \leq \nu < \alpha^*$. Since there is a regular α-cardinal $\eta \leq \alpha^*$ such that $\nu < \eta$ and $\xi < \eta$, $\bigcup_{j \in J}(\overline{M}_\lambda \cap \overline{C}^{\xi}_{j,\lambda} \cap [y])$ must have α-cardinality $< \eta \leq \alpha^*$, hence order type $< \alpha^*$. Thus $\overline{M}$ must have order type $< \alpha^*$. But $\overline{M}$ has order type $\geq \alpha^*$ by Lemma 3.2 and Lemma 2.6. This contradiction proves the lemma.

We now complete the proof of Theorem 4.2. By Lemma 4.9 and Lemma 4.10, $h\colon \omega \times \pi_2 \to \pi_2$ is such that for each $\xi < \pi_2$, $\lambda x h(x, \xi)\colon \omega \to \xi$ is an S_3 projection, and h is an S_3 function uniformly in $\xi < \pi_2$. By Lemma 4.8, $\text{s3cf}(\pi_2) = \omega$. Hence by Lemma 2.22 $\text{s3p}(\pi_2) = \omega$. Let $f_1\colon \pi_2 \to \alpha$ be a tame S_2 projection, and let $f_2\colon \omega \to \pi_2$ be an S_3 projection. By Lemma 2.14 and Lemma 2.19 $f_1 \circ f_2\colon \omega \to \alpha$ is an S_3 function, and is clearly a projection. Hence $\text{s3p}(\alpha) \leq \omega$. Since α is a limit ordinal, $\text{s3p}(\alpha) \geq \omega$, so $\text{s3p}(\alpha) = \omega$. This contradiction proves the theorem.

Theorem 4.11. *Let* $\kappa \in K$, $X \in \{A, B, R\}$. *If there exists a* $\kappa X \kappa X$*-maximal* α*-r.e. set, then* $\mathrm{s3p}(\alpha) = \omega$ *and* $\mathrm{s2cf}(\alpha) \leq \kappa$.

Proof. Assume first that there exists a $\kappa A \kappa A$-maximal α-r.e. set M. By (3.5) and (3.6), M is a κ-maximal α-r.e. set. Hence by Theorem 4.1, $\mathrm{s2cf}(\alpha) \leq \kappa$. If $\kappa \leq \alpha^*$, then by Theorem 4.2, $\mathrm{s3p}(\alpha) = \omega$. Assume $\kappa = \alpha$. If $\overline{M}$ is α-unbounded, then M is a $\kappa B \kappa B$-maximal α-r.e. set, so by Theorem 4.2, $\mathrm{s3p}(\alpha) = \omega$. Assume that $\overline{M}$ is α-bounded. By (3.5), M is an $\alpha A \alpha R$-maximal α-r.e. set, so by Lemma 3.3, M is an $\alpha^* A \alpha^* R$-maximal α-r.e. set. By (3.6), M is an $\alpha^* A \alpha^* B$-maximal α-r.e. set, so $\mathrm{s3p}(\alpha) = \omega$ by Theorem 2.2.

Assume next that there exists a $\kappa R \kappa R$-maximal α-r.e. set M. By (3.3) and (3.6), M is a κ-maximal α-r.e. set, so by Theorem 4.1, $\mathrm{s2cf}(\alpha) \leq \kappa$. If $\kappa \leq \alpha^*$, then by Theorem 4.2, $\mathrm{s3p}(\alpha) = \omega$. Assume $\kappa = \alpha$. If $\overline{M}$ is α-unbounded, then M is a $\kappa B \kappa B$-maximal α-r.e. set, so by Theorem 4.2, $\mathrm{s3p}(\alpha) = \omega$. Assume that $\overline{M}$ is α-bounded. By (3.3), M is a $\alpha A \alpha R$-maximal α-r.e. set, so by Lemma 3.3, M is an $\alpha^* A \alpha^* R$-maximal α-r.e. set. By (3.6) M is an $\alpha^* A \alpha^* B$-maximal α-r.e. set, so $\mathrm{s3p}(\alpha) = \omega$ by Theorem 2.2.

Assume, finally, that there exists a $\kappa B \kappa B$-maximal α-r.e. set M. By (3.4) and (3.3), M is a κ-maximal α-r.e. set, so by Theorem 4.1, $\mathrm{s2cf}(\alpha) \leq \kappa$. If $\kappa \leq \alpha^*$, then by Theorem 4.2, $\mathrm{s3p}(\alpha) = \omega$. But if $\kappa = \alpha$, then again by Theorem 4.2, $\mathrm{s3p}(\alpha) = \omega$. Thus the theorem is proved.

5. Existence. In this section, we construct xx-maximal α-r.e. sets for $x \in J$ and for certain admissible ordinals α. Combining this result with the nonexistence results of §4, we obtain a necessary and sufficient condition for the existence of xx-maximal α-r.e. sets in terms of cofinalities and projecta of α.

As we noted earlier, Friedberg [1] first constructed an $\omega A \omega A$-maximal α-r.e. set. His construction was generalized by Kreisel and Sacks [3] to get an $\omega B \omega A$-maximal α-r.e. set whenever $\alpha^* = \omega$. Lerman and Simpson [9] pushed this construction a step further to obtain an $\omega B \alpha A$-maximal α-r.e. set whenever $\mathrm{ts2p}(\alpha) = \omega$. Below we construct an $\alpha B \beta A$-maximal α-r.e. set whenever $\mathrm{s3p}(\alpha) = \omega$, where $\beta = \mathrm{s2cf}(\alpha)$.

Our construction differs from previous constructions in that we are forced to use an S_3 function in constructing maximal sets, whereas S_2 functions sufficed for previous constructions. To our knowledge, this is the first use of S_3 functions in priority argument constructions. They have subsequently appeared in Shore's [18] proof of the density theorem for the α-r.e. degrees. Sacks has been predicting constructions of this kind. He reasoned that just as S_2 functions are needed to generalize finite injury priority argument constructions, there should be a similar notion for infinite injury priority argument constructions. We use the

S_3 function to generate e-states, not priorities, but the function is of the type Sacks had in mind.

We feel that the best way of presenting the existence proof, is by defining certain functions, and presenting the construction and proof in terms of these functions. Unfortunately, much of the intuition is lost by describing the construction in this manner. Therefore, we will occasionally digress, and try to intuitively describe what is happening. We hope that this will aid the reader to follow the proof.

Theorem 5.1. *Let α be an admissible ordinal, and let $\beta = \mathrm{s2cf}(\alpha)$. Assume that $\mathrm{s3p}(\alpha) = \omega$. Then there exists an $\alpha B\beta A$-maximal α-r.e. set M.*

Proof. One should imagine the construction as taking place on $\alpha \times \omega$. We will refer to the point $\langle x, y\rangle \in \alpha \times \omega$ as *row* x, *column* y.

We will define a partial α-recursive function with α-recursive domain g': $\alpha \times \beta \to \alpha$. g: $\beta \to \alpha$ will be the partial function defined by $g(x) = \lim_{\sigma \to \alpha} g'(\sigma, x)$, for all $x < \beta$ such that $\lim_{\sigma \to \alpha} g'(\sigma, x)$ exists. Otherwise, $g(x)$ will be undefined. g will be strictly increasing on its domain, G, and $g(G)$ will be $\overline{M}$. As a result, $\overline{M}$ will have order type $\leq \beta$.

Since $\mathrm{s2cf}(\alpha) = \beta$, by Lemma 2.9, there is a strictly increasing tame S_2 cofinality function f: $\beta \to \alpha$. Let f': $\alpha \times \beta \to \alpha$ be an α-recursive function generating f as a tame S_2 function. f will pick out a strictly increasing sequence of rows of $\alpha \times \beta$, of order type β and cofinal with the rows of $\alpha \times \omega$. Elements of $\overline{M}$ will be associated with these rows in increasing order, at most one element with each row, so $\overline{M}$ will have order type $\leq \beta$.

Since $\mathrm{s3p}(\alpha) = \omega$, there is an S_3 projection h: $\omega \to \alpha$. Let h': $\alpha \times \alpha \times \omega \to \alpha$ be an α-recursive function generating h as an S_3 function. h' will serve two purposes in the construction. At each stage τ, h' is used to associate an α-r.e. set with each point of $\alpha \times \omega$. If $\langle \sigma, x\rangle \in \alpha \times \omega$, then $W_{h'(\sigma,\tau,x)}$ is associated with $\langle \sigma, x\rangle$ at stage τ. These α-r.e. sets are then used to determine e-states. h' is also used to pick out a column for row $f'(\tau, x)$ at stage τ. If a column is associated with row $f'(\tau, x)$ at stage τ, we call this column $\phi'(\tau, x)$. If no column is to be associated with row $f'(\tau, x)$ at stage τ, we may define $\phi'(\tau, x) = \omega$. Thus ϕ': $\alpha \times \beta \to \omega + 1$ will be a partial α-recursive function with α-recursive domain defined during the course of the construction. ϕ: $\beta \to \omega + 1$ defined by $\phi(x) = \lim_{\sigma \to \alpha} \phi'(\sigma, x)$ will have the property that for each $e < \omega$, $\phi^{-1}([e])$ is bounded below β. Hence for each $e < \omega$, we will eventually try to maximize the e-states of all sufficiently large elements of $\overline{M}$.

e-state functions were introduced by Friedberg [1] to construct maximal ω-r.e. sets, and all known constructions of maximal α-r.e. sets use some device similar to the e-state function. The e-state function we use depends on one more parameter than is the case with previous constructions of maximal sets, since we use a different e-state function for each row of $\alpha \times \omega$ which is in the range of f.

We define the e-state function E: $\beta \times \omega \times \alpha \times \alpha \to \omega$ by

$$E(\gamma, e, x, \sigma) = \sum\{2^{e-i}: i \leq e \text{ and } x \in W_{h'(f'(\sigma,\gamma),\sigma,i)}\}.$$

The following properties are standard for e-state functions, and are easy to verify.

(5.1) $m < n \Rightarrow [(E(\gamma, m, x, \sigma) < E(\gamma, m, y, \sigma)) \Rightarrow (E(\gamma, n, x, \sigma) < E(\gamma, n, y, \sigma))]$;

(5.2) if $\sigma < \tau$ and for all $i \leq e$, $h'(f'(\sigma, \gamma), \sigma, i) = h'(f'(\tau, \gamma), \tau, i)$, then $E(\gamma, e, x, \sigma) \leq E(\gamma, e, x, \tau)$; and

(5.3) $$E(\gamma, e, x, \sigma) < 2^{e+1}.$$

Let $\gamma < \beta$. We say that γ *requires attention at stage* σ if one of the following holds:

(5.4) $f'(\sigma, \gamma) \neq \lim_{\tau \to \sigma} f'(\tau, \gamma)$;

(5.5) $\lim_{\tau \to \sigma} \phi'(\tau, \gamma)$ is undefined;

(5.6) $\lim_{\tau \to \sigma} \phi'(\tau, \gamma)$ is defined, and for some $z \leq \lim_{\tau \to \sigma} \phi'(\tau, \gamma)$,

$$h'(f'(\sigma, \gamma), \sigma, z) \neq \lim_{\tau \to \sigma} h'(f'(\sigma, \gamma), \tau, z);$$

(5.7) $\lim_{\tau \to \sigma} \phi'(\tau, \gamma)$ is defined, and $\lim_{\tau \to \sigma} \phi'(\tau, \gamma) \neq \omega$, and $\lim_{\tau \to \sigma} g'(\tau, \gamma)$ is undefined;

(5.8) there are $x < \alpha$, $y < \alpha$, $e < \omega$, and ν such that

$$\gamma < \nu < \beta, \quad x = \lim_{\tau \to \sigma} g'(\tau, \sigma), \quad y = \lim_{\tau \to \sigma} g'(\tau, \nu),$$

$$e = \lim_{\tau \to \sigma} \phi'(\tau, \gamma), \quad \text{and} \quad E(\gamma, e, x, \sigma) < E(\gamma, e, y, \sigma).$$

The construction.

Stage 0. For all $\gamma < \beta$, $g'(\gamma, 0)$ and $\phi'(\gamma, 0)$ are undefined.

Stage $\sigma > 0$. Let γ be the least $\nu < \beta$ such that $\nu < \sigma$ and ν requires attention at stage σ. If no such γ exists, set $g'(\sigma, \nu) = \lim_{\tau \to \sigma} g'(\tau, \nu)$ for all

$\nu < \inf(\{\beta, \sigma\})$ such that $\lim_{\tau \to \sigma} g'(\tau, \nu)$ exists; otherwise, $g'(\sigma, \nu)$ is undefined Also, set $\phi'(\sigma, \nu) = \lim_{\tau \to \sigma} \phi'(\tau, \nu)$ for all $\nu < \inf(\{\beta, \sigma\})$ such that $\lim_{\tau \to \sigma} \phi'(\tau, \nu)$ exists; otherwise, $\phi'(\sigma, \nu)$ is undefined. Then go to the next stage.

If γ exists, we proceed by cases, and say that γ *receives attention at stage* σ.

Case 1. γ requires attention at stage σ through (5.4).

For each $\nu < \gamma$, define $g'(\sigma, \nu) = \lim_{\tau \to \sigma} g'(\tau, \nu)$ whenever $\lim_{\tau \to \sigma} g'(\tau, \nu)$ is defined. If $\nu \geq \gamma$ or $\lim_{\tau \to \sigma} g'(\tau, \nu)$ is undefined, then $g'(\sigma, \nu)$ is undefined. For each $\nu < \gamma$, define $\phi'(\sigma, \nu) = \lim_{\tau \to \sigma} \phi'(\tau, \nu)$ whenever $\lim_{\tau \to \sigma} \phi'(\tau, \nu)$ is defined. If $\nu \geq \gamma$ or $\lim_{\tau \to \sigma} \phi'(\tau, \nu)$ is undefined, then $\phi'(\sigma, \nu)$ is undefined. Then go to the next stage.

The ultimate purpose of the construction is to maximize the "e-state" of $g(\gamma)$ for all sufficiently large $\gamma < \beta$ and all $e < \omega$; by "e-state" here, we mean $E(\alpha) = \Sigma\{2^{e-i}: i \leq e \text{ and } x \in W_{b(i)}\}$. However, we only allow ourselves β rows to approximate b. Hence when f' changes value at stage σ, we need to use a new row, which gives us a new e-state function. We therefore erase all the mistakes we made, and start over again, leaving intact the work we have done which we still think may be correct.

Case 2. γ requires attention at stage σ through (5.5), but not through (5.4).

We define $g'(\sigma, \nu)$ as in Case 1, for all $\nu < \beta$. Also, define $\phi'(\sigma, \nu)$ as in Case 1 for all $\nu < \beta$ such that $\nu \neq \gamma$. If $\gamma = 0$, set $\phi'(\sigma, \gamma) = 0$. If $\gamma \neq 0$, $\phi'(\sigma, \gamma)$ is defined as follows. Let $e_1(\sigma)$ be the least $i < \omega$ such that $b'(f'(\sigma, \gamma), \sigma, i) \neq \lim_{\nu \to \gamma} b'(f'(\sigma, \nu), \sigma, i)$ if such an i exists. Otherwise, let $e_1(\sigma) = \omega$. Let $e_2(\sigma)$ be the least $i < \omega$ such that $\lim_{\nu \to \gamma} \phi'(\sigma, \nu) < i$ if $\lim_{\nu \to \gamma} \phi'(\sigma, \nu)$ exists and is less than ω. Otherwise, let $e_2(\sigma) = \omega$. Let $e(\sigma) = \inf(\{e_1(\sigma), e_2(\sigma)\})$. Set $\phi'(\sigma, \gamma) = e(\sigma)$. Then go to the next stage.

In this case, we think we have the correct row, but have not yet chosen an $e = e(\sigma)$ for which to maximize the e-state of $g(\gamma)$. We now choose such an e. We try to choose this e so that for each $i < \omega$, $\{\gamma < \beta: \lim_{\sigma \to \alpha} \phi'(\sigma, \gamma) = i\}$ is not cofinal with β. This allows us to maximize e-states for all $e < \omega$. b tells us when we have the correct e-state, so we can then go to the next e-state, and b' approximates to this point.

Case 3. γ requires attention at stage σ through (5.6), but not through (5.4) or (5.5).

Proceed exactly as in Case 2.

In this case, we find out that b' was a bad approximation to b for certain values z at previous stages, so we lick our wounds by cancelling the part of the construction dictated by the bad approximation, and prepare again to maximize the $e(\sigma)$-state of $g(\gamma)$, using a new guess at the values of b.

Case 4. γ requires attention at stage σ through (5.7), but not through (5.4), (5.5), or (5.6).

For all $\nu < \beta$, $\nu \neq \gamma$, define $g'(\sigma, \nu)$ as in Case 1. Set $g'(\sigma, \gamma) = \sigma$. For all $\nu < \beta$, $\nu \neq \gamma$, define $\phi'(\sigma, \nu)$ as in Case 1. Set $\phi'(\sigma, \gamma) = \lim_{\tau \to \sigma} \phi'(\tau, \gamma)$ if $\lim_{\tau \to \sigma} \phi'(\tau, \gamma)$ exists; otherwise, $\phi'(\sigma, \gamma)$ is undefined. Then go to the next stage.

In this case, we have fixed upon an e-state which $g(\gamma)$ should maximize, but do not yet have a candidate for $g(\gamma)$. We thus appoint σ as such a candidate.

Case 5. γ requires attention at stage σ through (5.8), but not through (5.4), (5.5), (5.6), or (5.7).

For all $\nu < \beta$, define $\phi'(\sigma, \nu)$ as in Case 4. For all $\nu < \beta$ such that $\nu \neq \gamma$, define $g'(\sigma, \nu)$ as in Case 1.

Let $x = \lim_{\tau \to \sigma} g'(\tau, \gamma)$ and $e = \lim_{\tau \to \sigma} \phi'(\tau, \gamma)$. Let ν be the least μ such that $\gamma < \mu < \beta$, $\lim_{\tau \to \sigma} g'(\tau, \mu)$ exists, and $E(\gamma, e, x, \sigma) < E(\gamma, e, \lim_{\tau \to \sigma} g'(\tau, \mu), \sigma)$. Let $y = \lim_{\tau \to \sigma} g'(\tau, \nu)$. Set $g'(\sigma, \gamma) = y$. Then go to the next stage.

In this case, we have found a candidate, y, for $g(\gamma)$, with greater e-state than that of the previous candidate, x. We thus make y the new candidate for $g(\gamma)$.

This completes the construction. Since the construction is α-effective, g and ϕ' are partial α-recursive functions with α-recursive domains. We define M by

$$x \in M \Leftrightarrow (\exists \sigma > x)(\nu < \sigma)(\nu < \beta \Rightarrow g'(\sigma, \nu) \neq x).$$

M is clearly an α-r.e. set.

For each $\gamma < \beta$, define

$$B_\gamma = \{\sigma: (\exists \nu \leq \gamma)(\nu \text{ requires attention at stage } \sigma)\},$$

and

$$B_{<\gamma} = \{\sigma: (\exists \nu < \gamma)(\nu \text{ requires attention at stage } \sigma)\}.$$

The following assertions are easy consequences of the construction. We leave them to the reader to verify.

(5.9) If γ requires attention at stage σ, then some $\nu \leq \gamma$ receives attention at stage σ.

(5.10) If $\sigma \geq \tau$ and $g'(\sigma, i)$ and $g'(\tau, i)$ are both defined, then $g'(\sigma, i) \geq g'(\tau, i)$.

(5.11) If $\sigma \geq \tau$, $g'(\sigma, i)$ and $g'(\tau, j)$ are both defined, and $g'(\sigma, i) = g'(\tau, j)$, then $i \leq j$.

As is the case with most constructions of maximal sets, there are three basic lemmas needed to prove Theorem 5.1: Lemma 5.8 which says that for each $\gamma < \beta$, B_γ is α-bounded; Lemma 5.14 which says that $\bar{M}$ is not αB-finite; and Lemma 5.18

which says that for each α-r.e. set C, either $\overline{M} \cap C$ or $\overline{M} \cap \overline{C}$ is βA-finite. In order to keep the proofs of all lemmas at a reasonable length, and since many facts proved along the way are used in the proofs of more than one of the basic lemmas, we will use a sequence of lemmas to prove the basic lemmas.

Lemmas 5.2 through 5.7 are proved with the intent of proving Lemma 5.8. We use an induction argument to show that B_γ is α-bounded. Assuming B_ν is α-bounded for all $\nu < \gamma$, we show that $B_{<\gamma}$ is α-bounded. We then show in succession, assuming $B_{<\gamma}$ is α-bounded, that γ requires attention through each of (5.4) through (5.8) α-boundedly often. This implies that B_γ is α-bounded.

Lemma 5.2. *Let $\gamma < \beta$ be given. Assume that for all $\nu < \gamma$, B_ν is α-bounded. Then $B_{<\gamma}$ is α-bounded.*

Proof. Define an α-recursive function ψ': $\alpha \times \gamma \to \alpha$ by $\psi'(\sigma, \nu) = \sup(B_\nu \upharpoonright_\sigma)$. Since B_ν is α-bounded for each $\nu < \gamma$, $\lim_{\sigma \to \alpha} \psi'(\sigma, \nu)$ exists for each $\nu < \gamma$. Let ψ: $\gamma \to \alpha$ be defined by $\psi(\nu) = \lim_{\sigma \to \alpha} \psi'(\sigma, \nu)$. Then ψ is an S_2 function. Since $\gamma < \beta = \mathrm{s2cf}(\alpha)$, $\sup(\psi([\gamma])) < \alpha$. But then $B_{<\gamma}$ is α-bounded, proving the lemma.

Lemma 5.3. *Let $\gamma < \beta$ and $\delta < \alpha$ be given such that* $\sup(B_{<\gamma}) \leq \delta$. *Then*

(5.12) *if $\nu < \gamma$ and $\sigma \geq \delta$, then $f'(\sigma, \nu) = f'(\delta, \nu)$. If, furthermore, for all $\sigma \geq \delta$, γ does not require attention at stage σ through* (5.4), *then for all $\sigma \geq \delta$, $f'(\sigma, \gamma) = f'(\delta, \gamma)$;*

(5.13) *if $\nu < \gamma$ and $\sigma \geq \delta$, then $\phi'(\sigma, \nu)$ is defined. If, furthermore, for all $\sigma \geq \delta$, γ does not require attention at stage σ through* (5.5), *then for all $\sigma \geq \delta$, $\phi'(\sigma, \gamma)$ is defined;*

(5.14) *if $\nu < \gamma$ and $\sigma \geq \delta$, then $\phi'(\sigma, \nu) = \phi'(\delta, \nu)$ and $h'(f'(\sigma, \nu), \sigma, e) = \lim_{\tau \to \sigma} h'(f'(\sigma, \nu), \tau, e)$ for all $e \leq \phi'(\delta, \nu)$. If, furthermore, for all $\sigma \geq \delta$, γ does not require attention at stage σ through* (5.4), (5.5), *or* (5.6), *then for all $\sigma > \delta$, $\phi'(\sigma, \gamma) = \phi'(\delta, \gamma)$ and $h'(f'(\sigma, \gamma), \sigma, e) = \lim_{\tau \to \sigma} h'(f'(\sigma, \gamma), \tau, e)$ for all $e \leq \phi'(\delta, \gamma)$;*

(5.15) *if $\nu < \gamma$, $\sigma \geq \delta$, and $\phi'(\delta, \nu) = e < \omega$, then $g'(\sigma, \nu)$ is defined. If $\nu < \gamma$ and $g'(\delta, \nu)$ is undefined, then $g'(\sigma, \nu)$ is undefined for all $\sigma \geq \delta$. If, furthermore, for all $\sigma \geq \delta$, γ does not require attention at stage σ through* (5.4), (5.5), (5.6), *or* (5.7), *and $\phi'(\delta, \gamma) = e < \omega$, then $g'(\sigma, \gamma)$ is defined for all $\sigma \geq \delta$; and if $g'(\delta, \gamma)$ is undefined, then $g'(\sigma, \gamma)$ is undefined for all $\sigma \geq \delta$; and*

(5.16) *if $\nu < \gamma$, $\sigma > \delta$, $\phi'(\delta, \nu)$ is defined, and $\phi'(\delta, \nu) = e < \omega$, then $g'(\sigma, \nu)$ is defined and $g'(\sigma, \nu) = g'(\delta, \nu)$. If, in addition, $\nu < \mu < \beta$ and $\lim_{\tau \to \sigma} g'(\tau, \mu)$ is defined, then $E(\nu, e, g'(\delta, \nu), \sigma) \geq E(\nu, e, \lim_{\tau \to \sigma} g'(\tau, \mu), \sigma)$.*

Proof. If $\nu \leq \gamma$, and σ is the least $\lambda \geq \delta$ such that $f'(\lambda, \nu) \neq f'(\delta, \nu)$, then

$f'(\sigma, \nu) \neq \lim_{\tau \to \sigma} f'(\tau, \nu) = f'(\delta, \nu)$. Hence ν requires attention at stage σ through (5.4), contradicting the hypothesis of the lemma. Thus (5.12) must be true.

If $\nu \leq \gamma$ and σ is the least $\lambda \geq \delta$ such that $\phi'(\lambda, \nu)$ is undefined, then $\lim_{\tau \to \sigma+1} \phi'(\tau, \nu)$ is undefined. Hence ν requires attention at stage $\sigma + 1$ through (5.5), contradicting the hypothesis of the lemma. Thus (5.13) must be true.

If $\nu \leq \gamma$ and $\sigma > \delta$, then by (5.13), $\phi'(\sigma, \nu)$ is defined. Assume that $\nu \leq \gamma$ and σ is the least $\lambda > \delta$ such that $\phi'(\lambda, \nu) \neq \phi'(\delta, \nu)$. Then $\lim_{\tau \to \sigma} \phi'(\tau, \nu) = \phi'(\delta, \nu)$. But by choice of σ, $\phi(\sigma, \nu)$ is defined by $\phi'(\sigma, \nu) = \lim_{\tau \to \sigma} \phi'(\tau, \nu)$. Thus we have a contradiction unless $\phi'(\sigma, \nu) = \phi'(\delta, \nu)$. Since $\phi'(\delta, \nu) = \lim_{\tau \to \sigma} \phi'(\tau, \nu)$ for all $\sigma > \delta$, if σ is the least $\lambda > \delta$ such that $h'(f'(\lambda, \nu), \lambda, e) \neq \lim_{\tau \to \lambda} h'(f'(\lambda, \nu), \tau, e)$ for some $e \leq \phi'(\delta, \nu)$, then ν requires attention at stage σ through (5.6), contradicting the hypothesis of the lemma. Hence (5.14) must be true.

If $\nu \leq \gamma$ and $\sigma \geq \delta$, then by (5.13), $\lim_{\tau \to \sigma+1} \phi'(\tau, \nu)$ is defined, and by (5.14), $\lim_{\tau \to \sigma+1} \phi'(\tau, \nu) = \phi'(\delta, \nu)$. Hence if $\phi'(\delta, \nu) = e < \omega$, then $g'(\sigma, \nu)$ must be defined, else ν would require attention at stage $\sigma + 1$ through (5.7), contradicting the hypothesis of the lemma. Furthermore, if $g'(\delta, \nu)$ is undefined, then for all $\sigma \geq \delta$, $g'(\sigma, \nu) = \lim_{\tau \to \sigma} g'(\tau, \nu) = g'(\delta, \nu)$. Hence if $g'(\delta, \nu)$ is undefined, $g'(\sigma, \nu)$ must be undefined for all $\sigma \geq \delta$. Thus (5.15) must be true.

Assume that $\nu < \gamma$, $\sigma > \delta$, $\phi'(\delta, \nu)$ is defined, and $\phi'(\delta, \nu) = e < \omega$. By (5.15), $g'(\sigma, \nu)$ is defined. For all τ such that $\delta < \tau \leq \sigma$, $g'(\tau, \nu)$ is defined by $g'(\tau, \nu) = \lim_{\lambda \to \tau} g'(\lambda, \nu)$ unless ν requires attention at stage τ through (5.7). In the latter case, since $\nu < \gamma$, we must have $\tau \in B_{<\gamma}$, hence $\tau < \gamma$, contradiction. Thus for all τ such that $\delta < \tau \leq \sigma$, $g'(\tau, \nu) = \lim_{\lambda \to \tau} g'(\lambda, \nu) = g'(\delta, \nu)$. In particular, $g'(\sigma, \nu) = g'(\delta, \nu)$. Next, assume also that $\nu < \mu < \beta$ and $\lim_{\tau \to \sigma} g'(\tau, \mu)$ is defined, but $E(\nu, e, g'(\delta, \nu), \sigma) < E(\nu, e, \lim_{\tau \to \sigma} g'(\tau, \mu), \sigma)$. By (5.12), $f'(\sigma, \nu) = f'(\delta, \nu) = \lim_{\tau \to \alpha} f'(\tau, \nu) = f(\nu)$, and by (5.14), for all $i \leq e$, $h'(f'(\sigma, \nu), \sigma, i) = \lim_{\tau \to \sigma} h'(f'(\sigma, \nu), \tau, i) = \lim_{\tau \to \sigma} h'(f(\nu), \tau, i)$. Also, for all $i \leq e$, $\lim_{\tau \to \sigma} h'(f(\nu), \tau, i) = h'(f(\nu), \delta + 1, i)$, else if σ is the least $\lambda > \delta$ such that $\lim_{\tau \to \lambda} h'(f(\nu), \tau, i) \neq h'(f(\nu), \delta + 1, i)$, then $\lim_{\tau \to \sigma} h'(f'(\sigma, \nu), \tau, i) = h'(f(\nu), \delta + 1, i) \neq h'(f'(\sigma, \nu), \sigma, i)$, hence ν will require attention at stage σ through (5.6), contradicting the hypothesis of the lemma. Also, by (5.14), $\lim_{\tau \to \sigma} g'(\tau, \nu) = g'(\delta, \nu)$. Hence ν requires attention at stage σ through (5.8), contradicting the hypothesis of the lemma. Thus (5.16) must be true. This concludes the proof of the lemma.

Lemma 5.4. *For each $\gamma < \beta$, there is a $\delta < \alpha$ such that for all σ, if $\delta \leq \sigma < \alpha$, then γ does not require attention at stage σ through* (5.4).

Proof. Since f' generates an S_2 function, there is a δ such that for all

$\sigma \geq \delta$, $f'(\sigma, \gamma) = f'(\delta, \gamma)$. Such a δ witnesses the truth of the lemma.

Lemma 5.5. *If $\nu < \omega$, then $\phi'(\sigma, \nu) < \omega$ for all σ such that $\phi'(\sigma, \nu)$ is defined.*

Proof. We show by induction on $\{\nu: \nu < \omega\}$ that if $\phi'(\sigma, \nu)$ is defined, then $\phi'(\sigma, \nu) \leq \nu$.

By induction, we can assume that $\phi'(\lambda, \mu) \leq \mu$ for all $\lambda \leq \sigma$ and $\mu < \nu$, or $\lambda < \sigma$ and $\mu = \nu$, whenever $\phi'(\lambda, \mu)$ is defined. $\phi'(\sigma, \nu)$ can be defined in one of three ways: Either $\phi'(\sigma, \nu) = \lim_{\tau \to \sigma} \phi'(\tau, \nu)$, so by induction, $\phi'(\sigma, \nu) \leq \nu$; or $\phi'(\sigma, \nu) = 0$ if $\nu = 0$ in which case $\phi'(\sigma, \nu) \leq \nu$; or $\phi'(\sigma, \nu) = e(\sigma)$ where $\nu > 0$ and $e(\sigma)$ is defined as in Case 2 of the construction. In the latter case, ν must receive attention at stage σ, $e(\sigma) \leq e_2(\sigma)$, and $e_2(\sigma) = \lim_{\mu \to \nu} \phi'(\sigma, \mu) + 1 = \phi'(\sigma, \nu - 1) + 1$. Note that $\phi'(\sigma, \nu - 1) = \lim_{\tau \to \sigma} \phi'(\tau, \nu - 1)$ and that $\lim_{\tau \to \sigma} \phi'(\tau, \nu - 1)$ is defined, else $\nu - 1$ requires attention at stage σ; so ν cannot receive attention at stage σ. By induction, $\phi'(\sigma, \nu - 1) \leq \nu - 1$, so $e_2(\sigma) \leq \nu$. This completes the proof of the lemma.

Lemma 5.6. *Let $\gamma < \beta$ be given, and assume that* $\sup(B_{<\gamma}) < \alpha$. *Then there is a stage $\delta < \alpha$ such that for all $\sigma > \delta$, γ does not require attention at stage σ through* (5.5) *or* (5.6).

Proof. Let $\delta_0 < \alpha$ be such that $\sup(B_{<\gamma}) \leq \delta_0$ and for all $\sigma \geq \delta_0$, γ does not require attention at stage σ through (5.4). δ_0 exists by Lemma 5.4. Let e_2 be the least $i < \omega$ such that $\lim_{\nu \to \gamma} \phi'(\delta_0, \nu) < i$ if such an i exists, and $e_2 = \omega$ otherwise. By (5.14), (5.9), and the definition of δ_0, if $\sigma > \delta_0$ and γ requires attention at stage σ through either (5.5) or (5.6), then γ receives attention at stage σ, the construction proceeds through either Case 2 or Case 3, and $e_2(\sigma) = e_2$. Let e_1 be the least $i < \omega$ such that $\lim_{\sigma \to \alpha} h'(f'(\delta_0, \gamma), \sigma, i) \neq \lim_{\nu \to \gamma} h'(f'(\delta_0, \nu), \delta_0, i)$, if such an i exists, and $e_1 = \omega$ otherwise. Let $e = \inf(\{e_1, e_2\})$.

Assume first that $e < \omega$. Let δ_1 be the least λ such that $\delta_0 < \lambda < \alpha$ and for all $\sigma \geq \lambda$ and $i \leq e$, $h'(f'(\lambda, \gamma), \sigma, i) = \lim_{\tau \to \alpha} h'(f'(\lambda, \gamma), \tau, i)$. Since $e < \omega$, the definition of h' implies that δ_1 exists. By (5.12) and (5.14), if $\sigma > \delta_1$ and γ requires attention at stage σ through (5.5) or (5.6), then $e_1(\sigma) = e_1$, hence $e(\sigma) = e = \phi'(\sigma, \gamma)$.

If there is no $\sigma \in B_\gamma$ such that $\sigma > \delta_1$ and γ requires attention at stage σ through (5.5) or (5.6), choose $\delta = \delta_1$ to prove the lemma. Otherwise, choose δ to be the least stage $\sigma \in B_\gamma$ such that $\sigma > \delta_1$ and γ requires attention at stage σ through (5.5) or (5.6). Then for all $\sigma > \delta$, $\phi'(\sigma, \gamma)$ is either defined by $\phi'(\sigma, \gamma) = \lim_{\tau \to \sigma} \phi'(\tau, \gamma)$, or $\phi'(\sigma, \gamma) = e(\sigma) = e$. In either case, $\phi'(\sigma, \gamma) = e$ for all $\sigma \geq \delta$, so for all $\sigma > \delta$, $\lim_{\tau \to \sigma} \phi'(\tau, \gamma)$ is defined. Hence if $\sigma > \delta$, γ cannot require attention at stage σ through (5.5). Since $\phi'(\sigma, \gamma) = e = \lim_{\tau \to \sigma}\phi'(\tau, \gamma)$ for all $\sigma > \delta$, by choice of $\delta \geq \delta_1$, γ cannot require attention at stage σ through (5.6) for any $\sigma > \delta$. Hence the lemma is proved if $e < \omega$.

Next assume that $e = \omega$. We define an α-recursive function ψ': $\alpha \times \omega \to \alpha$ as follows: If $\sigma \leq \delta_0$ and $i < \omega$, $\psi'(\sigma, i) = 0$; and if $\sigma > \delta_0$ and $i < \omega$, $\psi'(\sigma, i) = \sup(\{\lambda: \lambda < \sigma \text{ and } h'(f'(\delta_0, \gamma), \lambda, i) \neq h'(f'(\delta_0, \gamma), \sigma, i)\})$. By definition of h', $\lim_{\sigma \to \alpha} h'(f'(\delta_0, \gamma), \sigma, i)$ exists for each $i < \omega$, so $\lim_{\sigma \to \alpha}\psi'(\sigma, i)$ exists for each $i < \omega$. Define ψ: $\omega \to \alpha$ by $\psi(i) = \lim_{\sigma \to \alpha}\psi'(\sigma, i)$. ψ is an S_2 function. If $\gamma < \omega$, then $e_2 = \lim_{\nu \to \gamma}\phi'(\delta_0, \nu) + 1 = \phi'(\delta_0, \gamma - 1) + 1$ which is defined by (5.13), and $\phi'(\delta_0, \gamma - 1) < \omega$ by Lemma 5.5. Hence $e_2 < \omega$, so $e < \omega$ yielding a contradiction. Therefore, $\gamma \geq \omega$ and since $\text{s2cf}(\alpha) = \beta > \gamma$, $\text{s2cf}(\alpha) > \omega$. Hence $\psi([\omega])$ cannot be cofinal with α. Let $\delta_1 = \sup(\psi([\omega]) \cup \{\delta_0\})$. Then $\delta_1 < \alpha$ and for all $\sigma > \delta_1$ and all $i < \omega$, $h'(f'(\delta_0, \gamma), \sigma, i) = h'(f'(\delta_0, \gamma), \delta_1, i)$. Now if $\sigma > \delta_1$ and γ requires attention at stage σ through (5.5) or (5.6), since $\delta_1 \geq \delta_0$, γ will receive attention at stage σ, the construction will proceed through Case 2 or Case 3 at stage σ, and $e(\sigma) = e = \phi'(\sigma, \gamma)$. The proof is now completed exactly as in the preceding paragraph.

Lemma 5.7. *Let $\gamma < \beta$ be given, and assume that* $\sup(B_{<\gamma}) < \alpha$. *Then there is a stage $\delta < \alpha$ such that for all $\sigma > \delta$, γ does not require attention at stage σ through* (5.7) *or* (5.8).

Proof. Let $\delta_0 < \alpha$ be such that $\sup(B_{<\gamma}) \leq \delta_0$ and for all $\sigma \geq \delta_0$, γ does not require attention at stage σ through (5.4), (5.5), or (5.6). δ_0 exists by Lemma 5.4 and Lemma 5.6. By (5.9) and the choice of δ_0, if $\sigma \geq \delta_0$ and γ requires attention at stage σ, then γ receives attention at stage σ, and the construction proceeds through Case 4 or Case 5 at stage σ.

Let $B = B_\gamma \cap \{\sigma: \sigma \geq \delta_0\}$. If B is finite, then the lemma follows immediately. So assume for the sake of contradiction, that B is infinite. Let $\{\lambda_i: i < \omega\}$ be the first ω elements of B in order of magnitude. By (5.13) and (5.14), for all $\sigma \geq \delta_0$, $\phi'(\sigma, \gamma)$ is defined; and $\phi'(\sigma, \gamma) = \phi'(\delta_0, \gamma) = \lim_{\tau \to \sigma}\phi'(\tau, \gamma)$ for all $\sigma > \delta_0$. Hence for all i such that $0 < i < \omega$, $\lim_{\tau \to \lambda_i}\phi'(\tau, \gamma)$ is defined. Furthermore, $g'(\lambda_i, \gamma)$ is defined for all $i < \omega$ and if $\lambda_i < \sigma < \lambda_{i+1}$, $g'(\sigma, \gamma) = \lim_{\tau \to \sigma} g'(\tau, \gamma) = g'(\lambda_i, \gamma)$. Hence if $0 < i < \omega$, γ does not require attention at stage λ_i through (5.7), so γ must require attention at stage λ_i through (5.8). But then for each i such that $0 \leq i < \omega$, since $\lambda_i \geq \delta_0$, by (5.12), (5.13), and (5.14), $f'(\lambda_i, \gamma) = f(\gamma)$, $\phi'(\lambda_i, \gamma) = \phi'(\delta_0, \gamma) = \lim_{\tau \to \lambda_i}\phi'(\tau, \gamma)$, $h'(f'(\lambda_i, \gamma), \lambda_i, j) = h'(f(\gamma), \delta_0, j)$ for all $j \leq \phi'(\delta_0, \gamma)$, and $\phi'(\delta_0, \gamma) < \omega$. By Case 5 of the construction we must have $E(\gamma, \phi'(\delta_0, \gamma), g'(\lambda_i, \gamma), \lambda_{i+1}) < E(\gamma, \phi'(\delta_0, \gamma), g'(\lambda_{i+1}, \gamma), \lambda_{i+1})$. By (5.2),

$E(\gamma, \phi'(\delta_0, \gamma), g'(\lambda_{i+1}, \gamma), \lambda_{i+1}) \leq E(\gamma, \phi'(\delta_0, \gamma), g'(\lambda_{i+1}, \gamma), \lambda_{i+2})$. Thus $\{E(\gamma, \phi'(\delta_0, \gamma), g'(\lambda_i, \gamma), \lambda_{i+1}): i < \omega\}$ is infinite, and $\phi'(\delta_0, \gamma) < \omega$, contradicting (5.3). This completes the proof of the lemma.

Lemma 5.8. *For each $\gamma < \beta$, B_γ is α-bounded.*

Proof. The proof is a straightforward transfinite induction on $\{\gamma: \gamma < \beta\}$ using Lemma 5.2, Lemma 5.4, Lemma 5.6, and Lemma 5.7.

We define the partial function $g: \beta \to \alpha$ by $g(x) = \lim_{\sigma \to \alpha} g'(\sigma, x)$. Similarly, we define $\phi: \beta \to \omega + 1$ by $\phi(x) = \lim_{\sigma \to \alpha} \phi'(\sigma, x)$.

Our next basic lemma is Lemma 5.14, which says that $\overline{M}$ is not αB-finite. Lemmas 5.9 through 5.13 are aimed at proving this lemma. The proof is almost immediate from the following two facts: The range of g is cofinal with α; and $\overline{M}$ is exactly the range of g. The other lemmas provide some necessary information about g.

Lemma 5.9. *If $g(\gamma)$ is undefined, then there is a $\delta < \alpha$ such that for all $\sigma \geq \delta$, $g'(\sigma, \gamma)$ is undefined. Furthermore, ϕ is total on $[\beta]$.*

Proof. Let $\gamma < \beta$ be given. By Lemma 5.8, there is a $\delta < \alpha$ such that $\sup(B_\gamma) \leq \delta$. Fix such a δ. Note that $B_{<\gamma+1} = B_\gamma$.

For all $\sigma \geq \delta$, $g'(\sigma, \gamma) = \lim_{\tau \to \sigma} g'(\tau, \gamma)$. Hence by (5.16), if $g'(\delta, \gamma)$ is defined, then $g'(\sigma, \gamma) = g'(\delta, \gamma)$ for all $\sigma \geq \delta$. In this case, $g(\gamma) = \lim_{\sigma \to \alpha} g'(\sigma, \gamma) = g'(\delta, \gamma)$ is defined. Therefore $g'(\delta, \gamma)$ must be undefined. But then for all $\sigma \geq \delta$, $g'(\sigma, \gamma) = \lim_{\tau \to \sigma} g'(\tau, \gamma)$, so $g'(\sigma, \gamma)$ must be undefined. This proves the first part of the lemma.

By (5.13), $\phi'(\delta, \gamma)$ is defined, and by (5.14), $\phi'(\sigma, \gamma) = \phi'(\delta, \gamma)$ for all $\sigma \geq \delta$. Thus $\phi(\gamma) = \lim_{\sigma \to \alpha} \phi'(\sigma, \gamma) = \phi'(\delta, \gamma)$, completing the proof of the lemma.

Lemma 5.10. dom(g) *is cofinal with β.*

Proof. We assume, for the sake of contradiction, that the lemma is false. Then there is a $\gamma < \beta$ such that for all ν, if $\gamma \leq \nu < \beta$ then $g(\nu)$ is undefined.

Fix such a γ, and fix ν such that $\gamma \leq \nu < \beta$. We first show that $\phi(\nu) = \omega$. (Note that by Lemma 5.9, $\phi(\nu)$ is defined.) Assume that $\phi(\nu) = e < \omega$. Let $\delta < \alpha$ be such that $\sup(B_\nu) \leq \delta$. δ exists by Lemma 5.8. By (5.13) and (5.14), $\phi'(\delta, \nu)$ is defined, and $\phi'(\sigma, \nu) = \phi'(\delta, \nu) = \phi(\nu) = e$ for all $\sigma \geq \delta$. Hence if $\sigma > \delta$, $\lim_{\tau \to \sigma} \phi'(\tau, \nu) = e$. Let $\delta_1 > \delta$ be such that $g'(\delta_1, \nu)$ is undefined. δ_1 exists by Lemma 5.9. We now note that ν requires attention at stage δ_1 through (5.7), contradicting the choice of $\delta_1 > \delta$. We must therefore conclude that $\phi(\nu) = \omega$, and so $\phi'(\sigma, \nu) = \omega$ for all sufficiently large σ.

We next show that for all $i < \omega$, $h(i) = \lim_{\sigma \to \alpha} h'(f(\gamma), \sigma, i)$. For assume not.

Let i be the least $j < \omega$ such that $h(j) \neq \lim_{\sigma\to\alpha} h'(f(\gamma), \sigma, j)$. Since the range of f is cofinal with α, $h(i) = \lim_{\tau\to\alpha}\lim_{\sigma\to\alpha} h'(\tau, \sigma, i) = \lim_{\nu\to\beta}\lim_{\sigma\to\alpha} h'(f(\nu), \sigma, i)$. Thus there is a ν such that $\gamma < \nu < \beta$ and $\lim_{\sigma\to\alpha} h'(f(\nu), \sigma, i) \neq \lim_{\sigma\to\alpha} h'(f(\gamma), \sigma, i)$. Fix the least such ν. Let ξ_0 be the least $\xi < \alpha$ such that $\sup(B_{<\nu}) < \xi$ and such that if ν requires attention at stage σ through (5.4), then $\sigma < \xi$. ξ_0 exists by Lemma 5.8. Also, $\{\sigma < \xi_0: \phi'(\sigma, \nu) \text{ is undefined}\}$ is cofinal with ξ_0, so $\lim_{\tau\to\xi_0} \phi'(\tau, \nu)$ is undefined. Let ξ_1 be the least $\xi < \alpha$ such that $h'(f(\nu), \xi, j) = \lim_{\sigma\to\alpha} h'(f(\nu), \sigma, j)$ for all $j \leq i$. Since $i < \omega$, ξ_1 must exist by the definition of h'. By choice of γ and ξ_0, and by (5.12) and (5.14), if $\sigma \geq \xi_0$ and $j \leq i$, then $\lim_{\mu\to\nu} h'(f'(\sigma, \mu), \sigma, j) = \lim_{\mu\to\nu} h'(f(\mu), \sigma, j) = \lim_{\mu\to\nu}\lim_{\sigma\to\alpha} h'(f(\mu), \sigma, j) = h'(f(\gamma), \xi_0, j)$. Let $\xi = \max(\{\xi_0, \xi_1\})$. If $\xi = \xi_0$, then since $\lim_{\tau\to\xi} \phi'(\tau, \nu)$ is undefined, ν requires attention at stage ξ through (5.5); and if $\xi = \xi_1 > \xi_0$ since $h'(f'(\xi, \nu), \xi, i) \neq \lim_{\tau\to\xi} h'(f'(\xi, \nu), \tau, i)$, either ν requires attention at stage ξ through (5.5), or ν requires attention at stage ξ through (5.6), or $\lim_{\tau\to\xi} \phi'(\tau, \nu)$ is defined and $\lim_{\tau\to\xi} \phi'(\tau, \nu) < i$. In either of the first two cases, $\phi'(\xi, \nu)$ is defined by Case 2 or Case 3 of the construction; $\phi'(\xi, \nu) = e(\xi) \leq e_1(\xi) = i$, hence $\phi'(\xi, \nu) < \omega$. In the third case, $\phi'(\xi, \nu) = \lim_{\tau\to\xi} \phi'(\tau, \nu) < i < \omega$. By induction on $\{\sigma: \xi \leq \sigma < \alpha\}$, since $h'(f'(\sigma, \nu), \sigma, j) = \lim_{\tau\to\sigma} h'(f'(\sigma, \nu), \tau, j)$ for all $j \leq i$ and $\sigma > \xi$, $\phi'(\sigma, \nu) = \lim_{\tau\to\sigma} \phi'(\tau, \nu) = \phi'(\xi, \nu) < \omega$, for all σ such that $\xi < \sigma < \alpha$. Hence $\phi(\nu) = \phi'(\xi, \nu) < \omega$, contradicting the fact that $\phi(\nu) = \omega$ for all ν such that $\gamma \leq \nu < \alpha$. Thus for all $i < \omega$, $h(i) = \lim_{\sigma\to\alpha} h'(f(\gamma), \sigma, i)$.

Let δ_2 be such that $\sup(B_\gamma) \leq \delta_2 < \alpha$. δ_2 exists by Lemma 5.8. By (5.14), $\phi'(\delta_2, \gamma) = \phi(\gamma) = \omega$, and $h'(f(\gamma), \delta_2, i) = h(i)$ for all $i < \omega$. Hence h is an α-recursive function. Since α is admissible, and since $h([\omega]) = \alpha$, we must have $\alpha = \omega$. Hence $\beta = \text{s2cf}(\alpha) = \omega$, and since $\gamma < \beta$, $\gamma < \omega$. By Lemma 5.5, $\phi'(\sigma, \gamma) < \omega$ for all σ such that $\phi'(\sigma, \gamma)$ is defined. But since $\phi(\gamma)$ is defined and $\phi(\gamma) = \lim_{\sigma\to\alpha} \phi'(\sigma, \gamma)$, $\phi(\gamma) < \omega$. This contradicts the fact that we previously showed that $\phi(\gamma) = \omega$, and completes the proof of the lemma.

Lemma 5.11. *$\bigcup\{B_\mu: \mu < \beta\}$ is cofinal with α.*

Proof. Let $\xi < \alpha$ be given. If no $\gamma < \beta$ requires attention at any stage $\sigma \geq \xi$ through (5.4), then $f(\nu) = f'(\xi, \nu)$ for all $\nu < \beta$, hence f is an α-recursive function. Since α is admissible and f is a cofinality function, we must have $\beta = \alpha$. Thus there is a $\gamma < \beta$ such that $\xi < \gamma$.

At most one $\nu < \beta$ receives attention at any given stage σ, and the least ν which requires attention at stage σ receives attention at stage σ. Also, $\phi'(\sigma, \nu)$ cannot be defined if ν has not received attention at any stage $\delta \leq \sigma$, for $\phi'(0, \nu)$

is unaefined and for all $\delta \leq \sigma$, either $\phi'(\delta, \nu)$ is undefined, or $\phi'(\delta, \nu) = \lim_{\tau \to \delta} \phi'(\tau, \nu)$ which, by induction, is undefined. By the construction, γ cannot receive attention before stage γ. Hence γ requires attention at stage γ through (5.5). Thus $\gamma \in \bigcup \{B_\mu : \mu < \beta\}$ and $\xi < \gamma < \alpha$, proving the lemma.

Lemma 5.12. *The range of g is cofinal with* α.

Proof. Let $\delta < \alpha$ be given. By Lemma 5.11, there are $\gamma < \beta$ and λ such that $\delta < \lambda < \alpha$, and γ requires attention at stage λ. By (5.9), some $\mu \leq \gamma$ receives attention at stage λ. Fix such a μ and λ. Let ν be such that $\mu < \nu < \beta$ and $g(\nu)$ is defined. Such a ν exists by Lemma 5.10. Fix such a ν. Then $g'(\lambda, \nu)$ is undefined. Let σ be the least stage such that $\lambda < \sigma < \alpha$ and $g'(\sigma, \nu)$ is defined. σ must exist since $\lim_{\tau \to \alpha} g'(\tau, \nu) = g(\nu)$ is defined. Since $\lim_{\tau \to \sigma} g'(\tau, \nu)$ is undefined, $g'(\sigma, \nu)$ can only be defined by $g'(\sigma, \nu) = \sigma$. By (5.10), if $\tau \geq \sigma$ and $g'(\tau, \nu)$ is defined, then $g'(\tau, \nu) \geq g'(\sigma, \nu) = \sigma$. Hence $g(\nu) = \lim_{\tau \to \alpha} g'(\tau, \nu) \geq \sigma$. Since $\delta < \lambda < \sigma < \alpha$, $g(\nu) > \delta$, completing the proof of the lemma.

Lemma 5.13. $x \in \overline{M} \Leftrightarrow x$ *is in the range of g.*

Proof. Assume first that $x \in \overline{M}$. Then for all $\sigma \geq x$, x is in the range of $\lambda y g'(\sigma, y)$. Let $C = \{y : (\exists \sigma \geq x)(g'(\sigma, y) = x)\}$. By (5.11), if $y, z \in C$, $y \neq z$, $x \leq \sigma < \tau$, and $g'(\sigma, y) = x = g'(\tau, z)$, then $z < y$. Hence C must be finite, else we could construct an infinite decreasing sequence of ordinals. Let y be the least element of C, and let $\sigma \geq x$ be such that $g'(\sigma, y) = x$. Then $g'(\tau, y) = x$ for all $\tau \geq \sigma$, hence $g(y) = x$. Thus x is in the range of g.

Conversely, assume that $x \in M$. Then there is a $\sigma \geq x$ such that x is not in the range of $\lambda y g'(\sigma, y)$. We show that for all $\tau \geq \sigma$, x is not in the range of $\lambda y g'(\tau, y)$. Hence x cannot be in the range of g.

Assume, for the sake of contradiction, that for some δ such that $\sigma < \delta < \alpha$, x is in the range of $\lambda y g'(\delta, y)$. Fix the least such δ. Let z be such that $g'(\delta, z) = x$. Then for all τ such that $\sigma \leq \tau < \delta$, x is not in the range of $\lambda y g'(\tau, y)$, so $g'(\delta, z)$ cannot be defined by $g'(\delta, z) = \lim_{\tau \to \delta} g'(\tau, y)$ for any $\gamma < \beta$. Thus $g'(\delta, z)$ must be defined by Case 4 of the construction as $g'(\delta, z) = \delta$. Since $g'(\delta, z) = x$ and $x \leq \sigma < \delta$, we have the contradiction needed to prove the lemma.

Lemma 5.14. $\overline{M}$ *is not* αB*-finite.*

Proof. By Lemma 5.13, $\overline{M}$ is the range of g, and by Lemma 5.12, the range of g is α-unbounded. Hence $\overline{M}$ is α-unbounded, so $\overline{M}$ cannot be αB-finite.

Our final basic lemma, Lemma 5.18, says that for all α-r.e. sets C, either $\overline{M} \cap C$ or $\overline{M} \cap \overline{C}$ is βA-finite. Lemmas 5.15 through 5.17 are aimed at proving this lemma. The main idea is to show that for each $e < \omega$, $\phi^{-1}([e])$ is not cofinal with β. It will then follow that b' is a sufficiently good approximation to b to allow the proof to proceed along the standard lines of proofs of the existence of maximal sets.

Lemma 5.15. *If $\gamma < \nu < \beta$ and $g(\gamma)$ and $g(\nu)$ are both defined, then $g(\gamma) < g(\nu)$.*

Proof. It is easy to see by induction that for all σ, if $g'(\sigma, \gamma)$ and $g'(\sigma, \nu)$ are both defined and $\gamma < \nu$, then $g'(\sigma, \gamma) < g'(\sigma, \nu)$. But there is a $\sigma < \alpha$ such that for all $\tau \geq \sigma$, $g'(\tau, \gamma) = g'(\sigma, \gamma)$ and $g'(\tau, \nu) = g'(\sigma, \nu)$. Fix such a σ. Then $g(\gamma) = \lim_{\tau \to \alpha} g'(\tau, \gamma) = g'(\sigma, \gamma) < g'(\sigma, \nu) = \lim_{\tau \to \alpha} g'(\tau, \nu) = g(\nu)$.

Lemma 5.16. *Let $e < \omega$ be given. Then there is a $\gamma < \beta$ such that for all ν, if $\gamma \leq \nu < \beta$ then $\phi(\nu) \geq e$.*

Proof. By Lemma 5.9, $\phi(\nu)$ is defined for all ν such that $\gamma \leq \nu < \beta$. The proof is by induction on $\{e: e < \omega\}$. Assume as an induction hypothesis, that $\gamma < \beta$ is such that for all ν such that $\gamma \leq \nu < \beta$, $\phi(\nu) \geq e - 1$, if $e \neq 0$. If $e = 0$, the lemma is immediate.

By the definition of b', there is a $\lambda < \alpha$ such that for all $i \leq e$ and all σ such that $\lambda \leq \sigma < \alpha$, $\lim_{\tau \to \alpha} b'(\sigma, \tau, i) = b(i)$. Fix such a λ. Since $f: \beta \to \alpha$ is a cofinality function, there is a ν such that $\gamma < \nu < \beta$ and $f(\nu) \geq \lambda$. Fix such a ν. We show that for all μ such that $\nu < \mu < \beta$, $\phi(\mu) \geq e$.

Assume, for the sake of contradiction, that $\phi(\mu) < e$ for some μ such that $\nu < \mu < \beta$. Fix the least such μ. Since f is strictly increasing, $f(\mu) > \lambda$. Let δ_0 be the least δ such that $\sup(B_{<\mu}) < \delta$ and if μ requires attention at stage σ through (5.4), then $\sigma < \delta$. δ_0 exists by Lemma 5.8. By (5.12), (5.13), and (5.14), for all $\sigma \geq \delta_0$, $f'(\sigma, \mu) = f(\mu)$; and for all $\sigma \geq \delta_0$ and $\xi < \mu$, $\phi'(\sigma, \xi) = \phi'(\delta_0, \xi)$ and $b'(f(\xi), \sigma, i) = b'(f(\xi), \delta_0, i)$ for all $i \leq \phi'(\delta_0, \xi)$. We note that $\phi'(\tau, \mu)$ is undefined for a set of stages $\tau < \delta_0$ cofinal with δ_0, so $\lim_{\tau \to \delta_0} \phi'(\tau, \mu)$ is undefined. Let δ_1 be the least stage δ such that $b'(f(\mu), \delta, i) = b'(f(\mu), \tau, i)$ for all $i < e$ and $\tau \geq \delta$. δ_1 exists by the definition of b'. Let $\delta = \max(\{\delta_0, \delta_1\})$. We note that if $\delta_0 \geq \delta_1$, μ requires attention at stage δ through (5.5), and if $\delta_0 < \delta_1$, then either μ requires attention at stage δ through (5.5), or μ requires attention at stage δ through (5.6), or $\lim_{\tau \to \delta} \phi'(\tau, \mu)$ is defined and $\lim_{\tau \to \delta} \phi'(\tau, \mu) < e - 1$.

Assume first that μ requires attention at stage δ through (5.5). Then μ receives attention at stage δ, and the construction proceeds through Case 2 at stage δ.

Since $\delta \geq \delta_0$, for all $\tau \geq \delta$, $\lim_{\xi \to \mu} \phi'(\tau, \xi) = \lim_{\xi \to \mu} \phi'(\delta, \xi) \geq e - 1$ if this limit exists, as a result of the induction hypothesis. Hence $e_2(\tau) \geq e - 1 + 1 = e$

whenever μ requires attention at stage τ through (5.5) or (5.6) and $\tau \geq \delta$. Also, by choice of μ, for all $\tau \geq \delta$ and $i < e$, by (5.14) and the induction hypothesis,

$$h'(f'(\tau, \mu), \tau, i) = h'(f(\mu), \tau, i) = h'(f(\mu), \delta, i) = \lim_{\xi \to \mu} \lim_{\tau \to \alpha} h'(f(\xi), \tau, i)$$
$$= \lim_{\xi \to \mu} h'(f(\xi), \delta, i) = \lim_{\xi \to \mu} h'(f'(\tau, \xi), \tau, i).$$

Hence if $\tau \geq \delta$ and μ requires attention at stage τ through (5.5) or (5.6), then $e_1(\tau) \geq e$. Hence for all $\tau \geq \delta$, either $\phi'(\tau, \mu) = e(\sigma) \geq e$, or $\phi'(\tau, \mu) = \lim_{\sigma \to \tau} \phi'(\sigma, \mu)$, so in any case, if $\tau \geq \delta$ and $\phi'(\tau, \mu)$ is defined, then $\phi'(\tau, \mu) \geq e$. Hence $\phi(\mu) \geq e$, yielding the desired contradiction.

Assume next that μ requires attention at stage δ through (5.6), but not through (5.5). Then μ receives attention at stage δ, and the construction proceeds through Case 3 at stage δ. We obtain a contradiction exactly as in the preceding paragraph.

Finally, assume that $\lim_{\tau \to \delta} \phi'(\tau, \mu) = j < e - 1$, and $h'(f'(\delta, \mu), \delta, i) = \lim_{\tau \to \delta} h'(f'(\delta, \mu), \tau, i)$ for all $i \leq j$. If the latter does not hold, then μ would require attention at stage δ through (5.6), and we would be done by the preceding paragraph. Since for all $\sigma \geq \delta$ and $i \leq j$, $h'(f'(\sigma, \mu), \sigma, i) = h'(f'(\sigma, \mu), \delta, i)$, for all $\sigma \geq \delta$ $\phi'(\sigma, \mu) = \lim_{\tau \to \sigma} \phi'(\tau, \mu) = \phi'(\delta, \mu)$. Hence $\phi(\mu) = \phi'(\delta, \mu) = \lim_{\tau \to \delta} \phi'(\tau, \mu) < e - 1$, contradicting the induction hypothesis. This final contradiction concludes the proof of the lemma.

Lemma 5.17. *For all $e < \omega$, there is a $\gamma < \beta$ such that $g(\gamma)$ is defined and for all ν such that $\gamma < \nu < \beta$, $g(\nu) \in W_{h(e)}$ if and only if $g(\gamma) \in W_{h(e)}$.*

Proof. By Lemma 5.16 and the definition of h', there is a $\gamma < \beta$ such that for all ν, if $\gamma \leq \nu < \beta$, then $\phi(\nu) \geq e$ and $\lim_{\sigma \to \alpha} h'(f(\nu), \sigma, i) = h(i)$ for all $i \leq e$. Fix $e < \omega$ and fix the least γ as in the preceding sentence for e. By induction, for all $i < e$, there is a ν such that $\gamma < \nu < \beta$ and whenever μ and ξ are such that $\nu < \mu < \beta$, $\nu < \xi < \beta$, and $g(\mu)$ and $g(\xi)$ are both defined, then $g(\mu) \in W_{h(i)} \Leftrightarrow g(\xi) \in W_{h(i)}$. Fix such a ν.

We assume, for the sake of contradiction, that we have a μ and ξ such that $\nu < \mu < \xi < \beta$, $g(\mu)$ and $g(\xi)$ are both defined, $g(\mu) \notin W_{h(e)}$ and $g(\xi) \in W_{h(e)}$. Let δ be a stage such that $\xi \leq \delta < \alpha$, $\sup(B_\xi) < \delta$ and for all $i \leq e$, $g(\mu) \in W_{h(i)} \Leftrightarrow g(\mu) \in W^{\delta}_{h(i)}$ and $g(\xi) \in W_{h(i)} \Leftrightarrow g(\xi) \in W^{\delta}_{h(i)}$. δ exists by Lemma 5.8 and since $W^{\delta}_{h(i)}$ is an α-recursive approximation to $W_{h(i)}$. By (5.16), $g'(\delta, \xi) = g(\xi)$ and $g'(\delta, \mu) = g(\mu)$. By (5.12), $f'(\delta, \xi) = f(\xi)$ and $f'(\delta, \mu) = f(\mu)$. By (5.13) and (5.14), and since $\nu < \mu < \xi$, $\phi'(\delta, \xi) = \phi(\xi) \geq e$, $\phi'(\delta, \mu) = \phi(\mu) \geq e$, and $h'(f'(\delta, \xi), \delta, i) = h(i) = h'(f'(\delta, \mu), \delta, i)$ for all $i \leq e$. Hence by choice of δ, $E(\mu, e, g(\mu), \delta) <$

$E(\mu, e, g(\xi), \delta)$. By (5.1), $E(\mu, \phi(\mu), g(\mu), \delta) < E(\mu, \phi(\mu), g(\xi), \delta)$, which contradicts (5.16). This completes the proof of the lemma.

Lemma 5.18. *If C is any α-r.e. set, then either $C \cap \overline{M}$ or $\overline{C} \cap \overline{M}$ is βA-finite.*

Proof. Let C be an α-r.e. set. Then $C = W_{h(e)}$ for some $e < \omega$ since h is a projection. By Lemma 5.15 and Lemma 5.17, either $C \cap \overline{M}$ or $\overline{C} \cap \overline{M}$ is α-bounded. Let $\delta < \alpha$ be the least bound for either $C \cap \overline{M}$ or $\overline{C} \cap \overline{M}$. By Lemma 5.12, there is a $\nu < \beta$ such that $g(\nu) > \delta$. By Lemma 5.15, $g(\mu) > \delta$ for all μ such that $\nu \leq \mu < \beta$. Hence by Lemma 5.13, $\overline{M}\restriction_\delta$ has order type $\leq \nu < \beta$. Let γ be the order type of $\overline{M}\restriction_\delta$. Note that $\gamma < \beta$. By Lemma 2.12, $\beta \leq \alpha^*$ so $\gamma < \beta < \alpha^*$. Hence $\overline{M}\restriction_\delta$ is α-finite by Lemma 2.6.

If $\sup(C \cap \overline{M}) = \delta$, then $C \cap \overline{M} = C \cap \overline{M}\restriction_\delta$, so $C \cap \overline{M}$ is an α-r.e. set. Since $C \cap \overline{M} \subseteq \overline{M}\restriction_\delta$, $C \cap \overline{M}$ has order type $\gamma < \beta \leq \alpha^*$. By Lemma 2.5, $C \cap \overline{M}$ is α-finite, so $C \cap \overline{M}$ is βA-finite.

If $\sup(\overline{C} \cap \overline{M}) = \delta$, then $\overline{C} \cap \overline{M} = \overline{C \cup M}$, and $C \cup M$ is an α-r.e. set. Since $\overline{C} \cap \overline{M}\restriction_\delta = \overline{C} \cap \overline{M}$, $\delta < \alpha$, and $\gamma < \beta \leq \alpha^*$, Lemma 2.6 tells us that $\overline{C} \cap \overline{M}$ is α-finite, so $\overline{C} \cap \overline{M}$ is βA-finite. This completes the proof of the lemma.

Since M is an α-r.e. set, Theorem 5.1 follows immediately from Lemma 5.14 and Lemma 5.18.

We are now ready to give the necessary and sufficient condition for the existence of maximal α-r.e. sets.

Theorem 5.19. *Let $\kappa \in K$ and $X \in \{A, B, R\}$. Then there exists a $\kappa X \kappa X$-maximal α-r.e. set if and only if* $\mathrm{s3p}(\alpha) = \omega$ *and* $\mathrm{s2cf}(\alpha) \leq \kappa$.

Proof. If there exists a $\kappa X \kappa X$-maximal α-r.e. set, then by Theorem 4.11, $\mathrm{s3p}(\alpha) = \omega$ and $\mathrm{s2cf}(\alpha) \leq \kappa$.

Assume that $\mathrm{s3p}(\alpha) = \omega$ and $\mathrm{s2cf}(\alpha) = \beta \leq \kappa$. By Theorem 5.1, there exists an $\alpha B \beta A$-maximal α-r.e. set M. Since $\kappa \leq \alpha$, by (3.1), M is a $\kappa B \beta A$-maximal α-r.e. set. Since $\kappa \geq \beta$, by (3.2), M is a $\kappa B \kappa A$-maximal α-r.e. set. By (3.4), M is a $\kappa R \kappa A$-maximal α-r.e. set, so by (3.3), M is a $\kappa A \kappa A$-maximal α-r.e. set. By (3.5), M is a $\kappa R \kappa R$-maximal α-r.e. set. By (3.5), M is a $\kappa B \kappa R$-maximal α-r.e. set so by (3.6), M is a $\kappa B \kappa B$-maximal α-r.e. set. This completes the proof of the theorem.

6. Concluding remarks and open questions. We have studied maximal α-r.e. sets from a lattice theoretic point of view, and have obtained a necessary and sufficient condition for their existence, for various definitions of maximal set.

A. Nerode once asked us if maximal ω-r.e. sets could be constructed using some method radically different from the usual e-state construction. As we view it, an e-state construction differs from the usual finite injury priority method construction in that one approximates to requirements and tries to satisfy these approximations instead of trying to satisfy the natural requirements. Sacks and Simpson [17] have shown that finite injury priority arguments can be done for all admissible ordinals. But the failure of maximal sets to exist for all admissible ordinals is directly a failure of the method of e-states. Our necessary and sufficient condition for maximal α-r.e. sets to exist shows that any other method for constructing maximal ω-r.e. sets must have the same drawback as the e-state method.

The failure of maximal α-r.e. sets to exist for all admissible ordinals α should not be construed as a failure of the priority argument technique to generalize. For when it can be used to obtain maximal α-r.e. sets, the priority argument is virtually the same as that discussed in [17] to solve Post's problem. This failure should rather be attributed to the impossibility of obtaining requirements to which a priority argument can be applied. When maximal sets can be constructed, the requirements are obtained by using e-states, and it is these e-states which cannot always be generalized.

The failure of the e-state method in generalizations, is already evident for $e = \omega$. In trying to maximize the ω-state of the xth element of $\overline{M}$ (M is a maximal α-r.e. set), one may try ω successive elements with successively increasing ω-states as candidates for the xth element of $\overline{M}$, and then have no candidate whose ω-state is greater than the ω-states of all the preceding candidates. We like to think of this as a failure of compactness. In particular, the principle that every finite set of integers has a greatest element is what is lacking when one replaces "finite" with "α-finite" and "integers" with "ordinals".

Along with further comments, we list some open questions below. These questions are primarily meant to reflect our interest in the relationship of maximal α-r.e. sets to $\mathfrak{S}(\alpha)$. There are also many questions about $\mathfrak{S}(\alpha)$ not related to maximal sets. Ultimately, we hope that the structure of $\mathfrak{S}(\alpha)$ and the decidability of Th($\mathfrak{S}(\alpha)$) will be determined. The first two questions, however, consider maximal sets from a different point of view.

Maximal ω-r.e. sets were used by Martin and Pour-El to construct a maximal ω-r.e. elementary theory.

(Q1) Can the results of [11] be extended to arbitrary admissible ordinals? If so, is there any preferred definition of maximal α-r.e. set which can be used to obtain this result?

There are many generalizations of ordinary recursion theory, of which α-recursion theory is just one. For each such generalization questions of a similar nature to those dealt with in this paper should be considered.

(Q2) For every generalized recursion theory, what is a necessary and sufficient condition for the existence of maximal sets? In particular, we would like to single out Kreisel's replacement of "α-r.e." with "s.i.i.d." for consideration.

In §2, we indicated why the $\{\Sigma_n: n < \omega\}$ hierarchy was not suitable for defining the necessary and sufficient condition for the existence of maximal α-r.e. sets. We introduced definitions for functions to be S_2 and S_3. We say that f is an S_1 function if f is a Σ_1 function, and for $3 < n < \omega$, we say that $f: \beta \to \alpha$ is an S_n function if there is a function $g: \alpha \times \beta \to \gamma$ such that g is uniformly an S_{n-1} function in its first coordinate, and such that for all $x < \beta$, $f(x) = \lim_{\sigma \to \alpha} g(\sigma, x)$. Thus we have a new hierarchy $\{S_n: n < \omega\}$. We are interested in the relationship between this hierarchy and the $\{\Sigma_n: n < \omega\}$ hierarchy. It is easy to see that if $f \in S_n$, then $f \in \bigcup(\{\Sigma_n: n < \omega\})$.

(Q3) If $f \in \Sigma_n$, is $f \in \bigcup(\{S_n: n < \omega\})$? More particularly, is $\Sigma_n = S_n$ for any $n > 2$?

$\mathfrak{E}(\alpha)$ was defined in the introduction. A good introduction to $\mathfrak{E}(\omega)$ can be found in Rogers [14]. Let $\mathfrak{A} \subseteq \mathfrak{E}(\alpha)$. We say that $\mathfrak{A}$ is *lattice theoretic* over $\mathfrak{E}(\alpha)$ if there is a formula $F(x)$, in the language of lattice theory, with one free variable x, such that if $C \in \mathfrak{E}(\alpha)$, then $C \in \mathfrak{A}$ if and only if $F(C)$.

In §3, we showed that $\mathfrak{F} = \{X: X \text{ is } \alpha^*A\text{-finite}\}$ is lattice theoretic over $\mathfrak{E}(\alpha)$.

(Q4) Is there any other definition of finiteness in §3 which gives rise to a class $\mathfrak{G} \neq \mathfrak{F}$ such that $\mathfrak{G}$ is lattice theoretic over $\mathfrak{E}(\alpha)$?

In §3, we showed that $\mathfrak{M} = \{M: M \text{ is an } \alpha^*A\alpha^*A\text{-maximal } \alpha\text{-r.e. set}\}$ is lattice theoretic over $\mathfrak{E}(\alpha)$.

(Q5) Is there any other definition of maximality in §3 which gives rise to a class $\mathfrak{G} \neq \mathfrak{M}$ such that $\mathfrak{G}$ is lattice theoretic over $\mathfrak{E}(\alpha)$?

Towards answering (Q4) and (Q5), we propose the following questions. Consider all the definitions of maximal α-r.e. sets in §3. For each such definition:

(Q6) What are the possible order types of maximal α-r.e. sets M with $\overline{M}$ α-unbounded?

(Q7) What are the possible order types of maximal α-r.e. sets M with $\overline{M}$ α-bounded?

(Q8) When do two different definitions give rise to the same class of maximal α-r.e. sets?

(Q9) What is a necessary and sufficient condition for the existence of a maximal α-r.e. set M with $\overline{M}$ α-bounded?(7)

(7) (Q6), (Q7), (Q8), and (Q9) have recently been answered by A. Leggett in her doctoral dissertation written at Yale University.

If C and B are any two α-r.e. non-α-finite sets, and $f: \alpha \to C$ and $g: \alpha \to B$ are one-one α-recursive functions enumerating C and B respectively, then the map $f(x) \leftrightarrow g(x)$ is a lattice isomorphism of $\{X: X \subseteq C$ and $X \in \mathfrak{S}(\alpha)\}$ and $\{X: X \subseteq B$ and $X \in \mathfrak{S}(\alpha)\}$. Hence, setting $B = [\alpha]$ there exists an $\alpha^*A\alpha^*A$-maximal α-r.e. subset of C if and only if there exists an $\alpha^*A\alpha^*A$-maximal α-r.e. set. Hence if the necessary and sufficient condition answering (Q9) is not the same as the condition in Theorem 5.19, one could define the α-finite sets for such α where the conditions give different answers for existence, by F is α-finite if F is α^*A-finite or F has no $\alpha^*A\alpha^*A$-maximal α-r.e. subset. Hence (Q9) may well be relevant to finding an answer to (Q4).

Call M a *maximal subset* of S under any given definition of maximality, by replacing every occurrence of M in the definition with $S - M$. Thus M is maximal in S if $S - M$ is not κX-finite and cannot be split into two non-κX-finite pieces by any α-r.e. set. A maximal set M is of *type* 1 if given any α-r.e. set S maximal in M, there is an α-r.e. set C such that $S = M \cap C$. Otherwise, M is of *type* 2. Lachlan [6] showed that every maximal ω-r.e. set is of type 1. Owings [13] showed that there are maximal sets of both types in Metarecursion Theory, but his proof is valid for all α such that $\alpha > \omega$ and $\alpha^* = \omega$.

(Q10) Given any α such that maximal α-r.e. sets exist, what types of maximal sets can exist?

In §3, we indicated that for various definitions of finiteness, the quotient $\mathfrak{S}(\alpha)/\text{finite}(\alpha)$ is a lattice. For $\alpha = \omega$, Lachlan [6] has shown that $\mathfrak{S}(\omega)$ and $\mathfrak{S}(\omega)/\text{finite}(\omega)$ have the same degree of decidability.

(Q11) For which definitions of finiteness can the above mentioned theorem of [6] be proved?

(Q12) When are two quotient lattices as above isomorphic ($\cong$)?

(Q13) When are two quotient lattices as above elementarily equivalent ($\equiv$)?

(Q14) When is the elementary theory of such a quotient lattice decidable?

For each admissible ordinal α, $\text{Th}(\mathfrak{S}(\alpha))$ is a constructibly countable subset of ω. Hence there can be at most $\aleph_1^L$ distinct theories $\text{Th}(\mathfrak{S}(\alpha))$ as α ranges over all admissible ordinals. Since the class of $\alpha^*A\alpha^*A$-maximal α-r.e. sets is lattice theoretic over $\mathfrak{S}(\alpha)$, Theorem 5.19 implies that there are at least two distinct theories as above.

(Q15) How many distinct theories $\text{Th}(\mathfrak{S}(\alpha))$ are there?

(Q16) If $\alpha \neq \beta$ and α and β are admissible ordinals, what is a necessary and sufficient condition for $\mathfrak{S}(\alpha) \cong \mathfrak{S}(\beta)$? Are there admissible ordinals $\alpha \neq \beta$ such that $\mathfrak{S}(\alpha) \cong \mathfrak{S}(\beta)$?

(Q17) If α and β are admissible ordinals such that $\alpha \neq \beta$, what is a necessary and sufficient condition for $\mathcal{E}(\alpha) \equiv \mathcal{E}(\beta)$? Note that since there are at most $\aleph_1^L$ distinct theories, there will be many admissible ordinals α and β such that $\alpha \neq \beta$ and $\mathcal{E}(\alpha) \equiv \mathcal{E}(\beta)$.

The remaining questions deal with the decidability of $\mathrm{Th}(\mathcal{E}(\alpha))$. r-maximal α-r.e. sets are lattice theoretic over $\mathcal{E}(\alpha)$, and Lerman and Simpson [9] have shown that none exist for some admissible ordinals α. Lachlan [6] and Robinson [14] have shown that r-maximal ω-r.e. sets exist.

(Q18) What is a necessary and sufficient condition for the existence of r-maximal α-r.e. sets?

Lachlan [6] has shown that the class of hyperhypersimple ω-r.e. sets is lattice theoretic over $\mathcal{E}(\omega)$. Use this lattice theoretic definition to define hyperhypersimple α-r.e. sets.

(Q19) What is a necessary and sufficient condition for the existence of hyperhypersimple α-r.e. sets?(8)

More generally, in reference to decidability,

(Q20) What is the Turing degree of $\mathrm{Th}(\mathcal{E}(\alpha))$ for α-admissible? In particular, for which α, if any, is $\mathrm{Th}(\mathcal{E}(\alpha))$ decidable?

(Q21) What is the Turing degree of $\bigcap(\{\mathrm{Th}(\mathcal{E}(\alpha)): \alpha\text{-admissible}\})$? Is this theory decidable?

One of the major open questions of ordinary recursion theory, is the decidability of $\mathrm{Th}(\mathcal{E}(\omega))$. $\mathrm{Th}(\mathcal{E}(\alpha))$ seems to have less structure than $\mathrm{Th}(\mathcal{E}(\omega))$ for some admissible ordinals α. In particular, for $\alpha = \aleph_1^L$, there are no maximal α-r.e. sets and no r-maximal α-r.e. sets. It is our feeling that the question of the decidability of $\mathrm{Th}(\mathcal{E}(\aleph_1^L))$ is easier than the same question for $\mathrm{Th}(\mathcal{E}(\omega))$. Answering the former question may yield some information on the latter question.

Soare [20] has recently shown that if M_1 and M_2 are any two maximal ω-r.e. sets, then there is an automorphism of $\mathcal{E}(\omega)$ carrying M_1 to M_2. In the case of $\aleph_1^L$, it is not known whether the simple $\aleph_1^L$-r.e. sets can be divided into two disjoint nonempty classes through a formula, lattice theoretic over $\mathcal{E}(\aleph_1^L)$.

(Q22) For which admissible ordinals α, is there a formula $F(x)$ lattice theoretic over $\mathcal{E}(\alpha)$ which divides the simple α-r.e. sets into two disjoint nonempty

(8) Recent results of S. B. Cooper, C. T. Chong, and M. Lerman have yielded both existence and nonexistence results for hyperhypersimple α-r.e. sets for a large class of admissible ordinals α. There are still some cases, however, where the question has not been resolved.

classes? In particular, is there such a formula for $\alpha = \aleph_1^L$?

If no hyperhypersimple $\aleph_1^L$-r.e. sets exist, then the following question seems reasonable.

(Q23) Let S_1 and S_2 be any two distinct simple $\aleph_1^L$-r.e. sets. Is there an automorphism of $\mathcal{E}(\aleph_1^L)$ carrying S_1 to S_2?

Finally, we return to Lachlan's [5] theorem mentioned in the introduction, that a certain class of sentences of $\mathrm{Th}(\mathcal{E}(\omega))$ is decidable over $\mathcal{E}(\omega)$. This theorem was extended by Machtey [10] for $\alpha^* = \omega$.

(Q24) For which admissible ordinals α is Lachlan's class of formulas decidable over $\mathcal{E}(\alpha)$?

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